Accuracy improvements and investigations of a compressible second order finite volume code towards the incompressible limit

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STAB Workshop 2015

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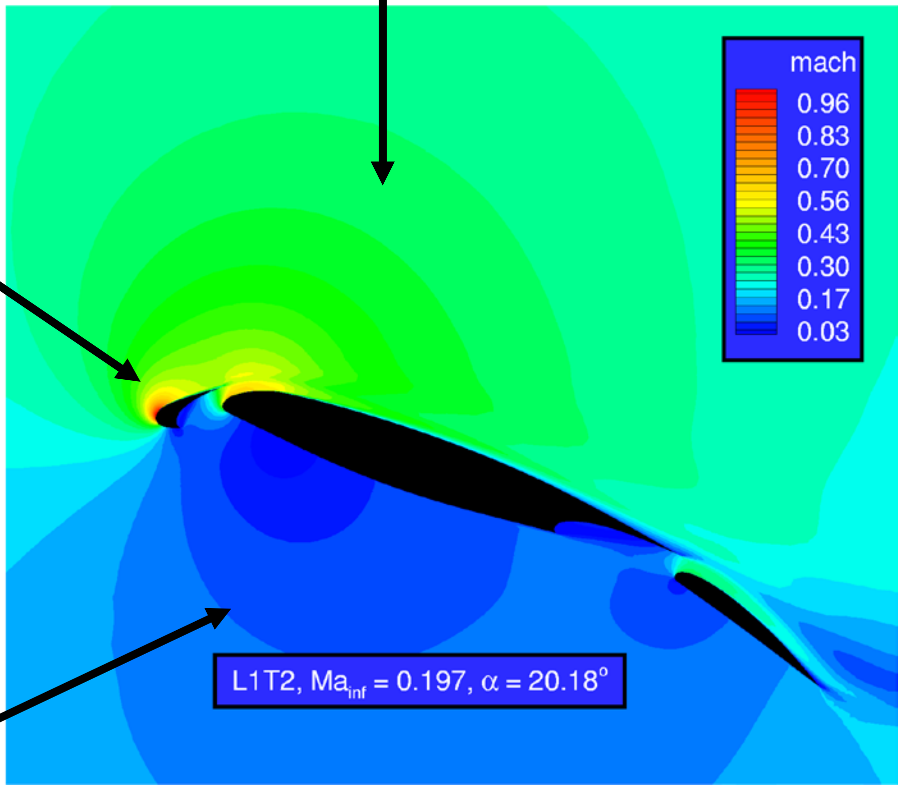
Importance for flow solver to treat low Mach number flows

Incompressible flow with
low Mach numbers

Local compressible
flow effects with large
Mach numbers



Incompressible flow with
low Mach numbers

Flow over a multi-element airfoil
at high angle of attack

Compressible Euler Equations

Incompressible Euler Equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$\longleftrightarrow$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + p)}{\partial x} +$$

Major Difference:

Time derivative of density in the continuity equation has disappeared

$$0$$

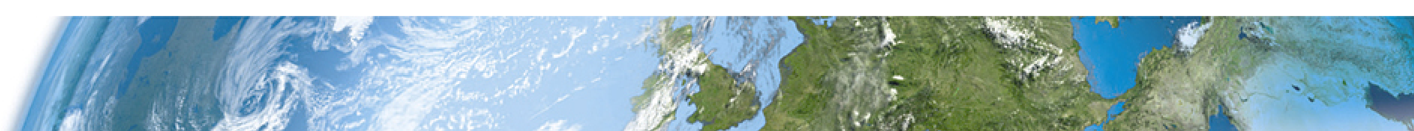
$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2 + p)}{\partial y} = 0$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2 + p)}{\partial y} = 0$$

$$\frac{\partial(\rho E)}{\partial t} + \frac{\partial(\rho u H)}{\partial x} + \frac{\partial(\rho v H)}{\partial y} = 0$$

Equation of state: $p = \rho R T$

Speed of sound: $\frac{dp}{d\rho} = a^2$



Compressible Euler Equations

Incompressible Euler Equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0 \quad \longleftrightarrow \quad \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0$$

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} + \frac{\partial (\cdots}{\cdots} \cdots 0$$

$$\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho u v)}{\partial x} + \frac{\partial (\rho v^2 \cdots}{\partial y} \cdots 0$$

$$\frac{\partial (\rho E)}{\partial t} + \frac{\partial (\rho u H)}{\partial x} + \frac{\partial (\rho v \cdots}{\partial y} \cdots$$

Major Difference:
Time derivative of density in the continuity equation has disappeared

This changes the character of the equations from purely hyperbolic in time to elliptic.

Equation of state: $p = \rho R T$

Speed of sound: $\frac{dp}{d\rho} = c^2$

Compressible Euler Equations

Incompressible Euler Equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0 \quad \longleftrightarrow \quad \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0$$

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} + \dots \quad 0$$

$$\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho u v)}{\partial x} + \frac{\partial (\rho v^2 \dots}{\partial y} \dots \quad 0$$

$$\frac{\partial (\rho E)}{\partial t} + \frac{\partial (\rho u H)}{\partial x} + \frac{\partial (\rho v \dots}{\partial y} \dots$$

Major Difference:

Time derivative of density in the continuity equation has disappeared

This changes the character of the equations from purely hyperbolic in time to elliptic, such that the velocity field satisfies an elliptic divergence constraint.

Equation of state: $p = \rho R T$

Speed of sound: $\frac{dp}{d\rho} = c^2$

Incompressible limit: Velocity field is divergence free

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Compressible Euler Equations

Incompressible Euler Equations

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0 \quad \longleftrightarrow \quad \frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0$$

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2+p)}{\partial x}+\dots \quad 0$$

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho u v)}{\partial x}+\frac{\partial(\rho v^2 \dots}{\partial y} \quad 0$$

$$\frac{\partial(\rho E)}{\partial t}+\frac{\partial(\rho u H)}{\partial x}+\frac{\partial(\rho v \dots}{\partial y}$$

Major Difference:

Time derivative of density in the continuity equation has disappeared

This changes the character of the equations from purely hyperbolic in time to elliptic, such that the velocity field satisfies an elliptic divergence constraint.

Equation of state: $p = \rho R T$

Incompressible limit: Velocity

→ Flow solvers which are based on either the compressible or the incompressible fluid flow equations are not suited to simulate flows of varying types

$$\frac{\ }{\partial x}+\frac{\ }{\partial y}=0$$

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Historical review for treatment of low Mach number flows

- Klainerman, S. & Majda, A.: „Singular limits of a quasilinear hyperbolic system with large parameters and the incompressible limit of compressible fluids“, Comm. Pure. Appl. Math., 1981
- Asano, K.: „On the Incompressible Limit of the Compressible Euler Equations“, Japan J. Appl. Math, 1987
- Turkel, E.: „Preconditioned Methods for Solving the Incompressible and Low Speed Compressible Equations“, Journal of Computation Physics, 1987
- Choi, D. & Merkle, C.L.: „The Application of Preconditioning in Viscous Flow“, Journal of Computational Physics, 1993
- Guillard, H. & Viozat, C.: „On the Behaviour of Upwind Schemes in the Low Mach Number Limit“, Journal of Computers & Fluids, 1999

  → Meister, A.: „Asymptotic Expansions and Numerical Methods in Computational Fluid Dynamics“, Hamburger Beiträge zur Angewandten Mathematik, 2001

  → **Overview and analysis of the methods**
- Rossow: „Extension of a Compressible Code Toward the Incompressible Limit“, AIAA Journal, 2003

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  → **Overview and analysis of the methods**
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Suggestion of Low Mach number preconditioner which seems to be implemented in several codes solving the compressible equations

It seems that not much attention has been drawn to this suggestion

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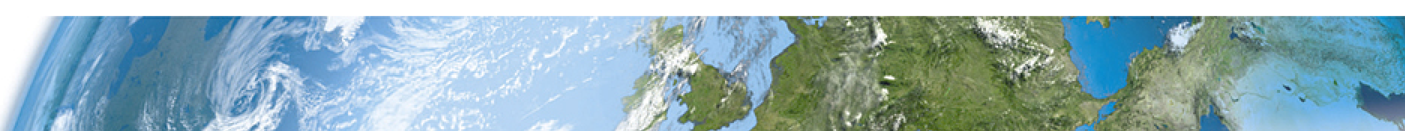

Discretization of convective terms: Roe scheme

$$\int_{\partial\Omega_i} (\mathrm{f_c} \bullet \mathrm{n})\,ds \approx \sum_{j\in N(i)} \frac{1}{2}\Big((\mathrm{f_c} \bullet \mathrm{n})(\mathrm{W}_i) + (\mathrm{f_c} \bullet \mathrm{n})(\mathrm{W}_j) - D_{ij}(\mathrm{W}) \Big)$$

$$D_{ij}(\mathrm{W}) = \left|\mathrm{A}_{ij}\right| (\Delta \mathrm{W}) \qquad \Delta \mathrm{W} \cong \begin{cases} \mathrm{O}(\mathrm{h}) & \text{second differences} \\ \mathrm{O}(\mathrm{h}^2) & \text{fourth differences} \end{cases}$$

$$\mathrm{A}_{ij} = \frac{\partial (\mathrm{f_c} \bullet \mathrm{n})}{\partial \mathrm{W}} \left[\mathrm{W_{Roe}}\right]$$

$$\left|\mathrm{A}_{ij}\right| = \mathrm{Q}\left|\Lambda\right|\mathrm{Q}^{-1}, \ (\mathrm{Q}, \Lambda) \ \text{eigendecomposition of } \mathrm{A}_{ij}$$

→ Discretization shows an excess of artificial viscosity for $\mathrm{Ma} \to 0$

→ Modification of upwinding is required

Low Mach modification (Turkel, Guillard & Viozat)

$$\int_{\partial\Omega_i} (\mathrm{f_c} \bullet \mathrm{n})\,\mathrm{ds} \approx \sum_{j \in N(i)} \frac{1}{2}\left((\mathrm{f_c} \bullet \mathrm{n})(\mathrm{W}_i) + (\mathrm{f_c} \bullet \mathrm{n})(\mathrm{W}_j) - D_{ij}(\mathrm{W})\right)$$

$$D_{ij}(\mathrm{W}) = \mathrm{P}_{ij}^{-1}\left|\mathrm{P}_{ij}\,\mathrm{A}\right|(\Delta\mathrm{W}) \qquad \Delta\mathrm{W} \cong \begin{cases} \mathrm{O(h)} & \text{second differences} \\ \mathrm{O(h^2)} & \text{fourth differences} \end{cases}$$

$$\mathrm{W}_{prim}^{(0)} = (p, u_1, u_2, u_3, S), \; S = \ln(p/\rho^{\gamma})$$

$$\mathrm{P_{LM}} = \begin{pmatrix} m & 0 & 0 & 0 & -m\delta \\ -\dfrac{\alpha u_1}{\rho a^2} & 1 & 0 & 0 & \dfrac{\alpha u_1}{\rho a^2}\delta \\ -\dfrac{\alpha u_2}{\rho a^2} & 0 & 1 & 0 & \dfrac{\alpha u_2}{\rho a^2}\delta \\ -\dfrac{\alpha u_3}{\rho a^2} & 0 & 0 & 1 & \dfrac{\alpha u_3}{\rho a^2}\delta \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Free parameters: $\alpha, \; \delta, \; m$

The role of α is not clear, and I don‘t know a paper where this parameter is investigated

☞ $\alpha = 0$

Low Mach modification (Turkel, Guillard & Viozat)

$$\int_{\partial\Omega_i} (\mathrm{f_c} \bullet \mathrm{n})\,\mathrm{ds} \approx \sum_{j \in N(i)} \frac{1}{2}\Big((\mathrm{f_c} \bullet \mathrm{n})(\mathrm{W}_i) + (\mathrm{f_c} \bullet \mathrm{n})(\mathrm{W}_j) - D_{ij}(\mathrm{W})\Big)$$

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$$\mathrm{W}_{prim}^{(0)} = (p, u_1, u_2, u_3, S), \; S = \ln(p / \rho^{\gamma})$$

$$\begin{pmatrix} m & 0 & 0 & 0 & -m\delta \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Free parameters: $\alpha,\ \delta,\ m$

Low Mach modification does not come out of the blue, but is based on asymptotic analysis and expansions

→ Beyond the scope of this talk, for details we refer to e.g.

→ Meister, A.: „Asymptotic Expansions and Numerical Methods in Computational Fluid Dynamics“

Low Mach modification (Turkel, Guillard & Viozat)

$$\int_{\partial\Omega_i} (\mathrm{f_c} \bullet \mathrm{n})\mathrm{ds} \approx \sum_{j \in N(i)} \frac{1}{2}\left((\mathrm{f_c} \bullet \mathrm{n})(\mathrm{W}_i) + (\mathrm{f_c} \bullet \mathrm{n})(\mathrm{W}_j) - D_{ij}(\mathrm{W})\right)$$

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$$\mathrm{W}_{prim}^{(0)} = (p, u_1, u_2, u_3, S), \ S = \ln(p/\rho^{\gamma})$$

$$\mathrm{P_{LM}} = \begin{pmatrix} m & 0 & 0 & 0 & -m\delta \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ [\mathrm{P_{LM}}]^{-1} = \begin{pmatrix} 1/m & 0 & 0 & 0 & \delta \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left|\mathrm{P}_{ij}\mathrm{A}_{ij}\right| = \mathrm{Q_{LM}}\left|\Lambda_{\mathrm{LM}}\right|\mathrm{Q}_{\mathrm{LM}}^{-1}, \ (\mathrm{Q_{LM}}, \Lambda_{\mathrm{LM}}), \text{ eigendecomposition of } \mathrm{P}_{ij}\mathrm{A}_{ij}$$

→ Reformulate in conservative variables

Parameter choice for Turkel's preconditioner

$$W_{prim}^{(0)} = (p, u_1, u_2, u_3, S),\ \ S = \ln(p/\rho^{\gamma})$$

$$P_{LM} = \begin{pmatrix} M_r & 0 & 0 & 0 & -M_r\delta \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ \ [P_{LM}]^{-1} = \begin{pmatrix} 1/M_r & 0 & 0 & 0 & \delta \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_r := \min\left\{ \max\left\{ \mathrm{Ma}^2, K\mathrm{Ma}_\infty^2 \right\},\ 1 \right\},\qquad K > 0$$

$$\delta := 0 \quad \text{or} \quad \delta := \begin{cases} 0, & Ma^2 \geq 1 \\ 1, & Ma^2 < 1 \end{cases}$$

→ Designed such that $P_{LM} = \mathrm{Id} \Leftrightarrow \mathrm{Ma} \geq 1$

Difference of Roe scheme and modified scheme of Turkel

Eigenvalues of non modified scheme:

$$\lambda_1 = \lambda_2 = \lambda_3 = V, \quad \lambda_4 = V + aA, \qquad \lambda_5 = V - aA$$

V = normal velocity

A = surface area

$$\text{Assumption}: a \sim O\left(\frac{1}{Ma}\right), \quad Ma \to 0$$

$$|A_{ij}| \sim \begin{pmatrix} O(1) & O(Ma) & O(Ma) & O(Ma) & O(Ma) \\ O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O(Ma) \\ O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O(Ma) \\ O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O(Ma) \\ O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) \end{pmatrix}$$

Represent $|A_{ij}|$ with respect to Mach number and insert asymptotic behavior of terms

$|A_{ij}|\Delta W \sim O\left(\frac{h}{Ma}\right)$ for the momentum equations→causes loss of accuracy

See Viozat, Implicit Upwind Schemes for Low Mach Number compressible flows, 1997

Difference of Roe scheme and modified scheme ... Turkel

Eigenvalues of non modified scheme:

$$\lambda_1 = \lambda_2 = \lambda_3 = V, \quad \lambda_4 = V + a, \quad \ldots a$$

$$\text{Assumption}: a \sim O\left(\frac{1}{\ldots}\right) \ldots \to 0$$

With respect to this analysis one can increase the degrees of freedom to deal with low Mach number flows

$$|A_{ij}| \sim \begin{pmatrix} O(1) & O(Ma) & O(Ma) & & O(Ma) \\ O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & & O\left(\frac{1}{Ma}\right) & O(Ma) \\ O\left(\frac{1}{Ma}\right) & & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O(Ma) \\ & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O(Ma) \\ & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) \end{pmatrix}$$

Represent $|A_{ij}|$ with respect to Mach number and insert asymptotic behavior of terms

$$|A_{ij}|\Delta W \sim O\left(\frac{h}{Ma}\right)$$ for the momentum equations→causes loss of accuracy

See Viozat, Implicit Upwind Schemes for Low Mach Number compressible flows, 1997

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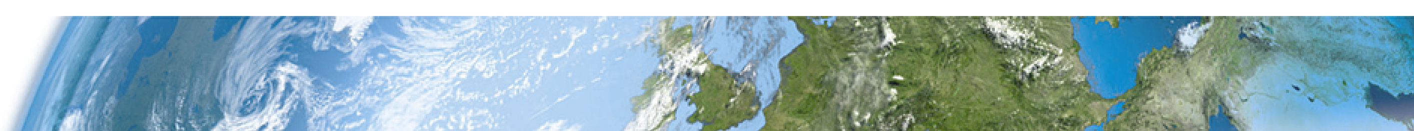

Difference of Roe scheme and modified scheme of Turkel

Eigenvalues of non modified scheme:

$$\lambda_1 = \lambda_2 = \lambda_3 = V, \quad \lambda_4 = V + a, \quad \lambda_5 = V - a$$

$$\text{Assumption}: a \sim O\left(\frac{1}{Ma}\right) \quad Ma \to 0$$

$$|A_{ij}| \sim \begin{pmatrix} O(1) & O(Ma) & O(Ma) & O(Ma) & O(Ma) \\ O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O(Ma) \\ O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O(Ma) \\ O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O(Ma) \\ O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) & O\left(\frac{1}{Ma}\right) \end{pmatrix}$$

Represent $|A_{ij}|$ with respect to Mach number and present asymptotic behavior of terms

$$|A_{ij}|\Delta W \sim O\left(\frac{h}{Ma}\right)$$ for the momentum equations → causes loss of accuracy

See Viozat, Implicit Upwind Schemes for Low Mach Number compressible flows, 1997

With respect to this analysis one can increase the degrees of freedom to deal with low Mach number flows

Then one decreases the onflow Mach number

Difference of modified scheme of Turkel

Eigenvalues of modified Turkel scheme:

$$\lambda_1 = \lambda_2 = \lambda_3 = \mathrm{V},\ \ \lambda_{4/5} = \frac{1}{2}\left(\mathrm{V} + \mathrm{M}_r \mathrm{V} \pm \mathrm{T}\right), \quad \mathrm{T} := \sqrt{(\mathrm{M}_r \mathrm{V} - \mathrm{V})^2 + 4\mathrm{M}_r \mathrm{A}^2 a^2}$$

$$\mathrm{M}_r = \min\left\{\max\left\{\mathrm{Ma}^2, K\mathrm{Ma}_\infty{}^2\right\}, 1\right\}, \quad K \geq 1$$

$$\mathrm{P}_{ij}^{-1}\left|\mathrm{P}_{ij}\mathrm{A}_{ij}\right| \sim \begin{pmatrix} \mathrm{O}(1) & \mathrm{O}(1) & \mathrm{O}(1) & \mathrm{O}(1) & \mathrm{O}(1) \\ \mathrm{O}(1) & \mathrm{O}(1) & \mathrm{O}(1) & \mathrm{O}(1) & \mathrm{O}(\mathrm{Ma}) \\ \mathrm{O}(1) & \mathrm{O}(1) & \mathrm{O}(1) & \mathrm{O}(1) & \mathrm{O}(\mathrm{Ma}) \\ \mathrm{O}(1) & \mathrm{O}(1) & \mathrm{O}(1) & \mathrm{O}(1) & \mathrm{O}(\mathrm{Ma}) \\ \mathrm{O}\left(\frac{1}{\mathrm{Ma}^2}\right) & \mathrm{O}\left(\frac{1}{\mathrm{Ma}^2}\right) & \mathrm{O}\left(\frac{1}{\mathrm{Ma}^2}\right) & \mathrm{O}\left(\frac{1}{\mathrm{Ma}^2}\right) & \mathrm{O}(1) \end{pmatrix}$$

Represent $\left|\mathrm{A}_{ij}\right|$ with respect to Mach number and insert asymptotic behavior of terms

$\left|\mathrm{A}_{ij}\right|\Delta\mathrm{W} \sim \mathrm{O}(\mathrm{h})$ for the momentum equations→ improved accuracy

See Viozat, Implicit Upwind Schemes for Low Mach Number compressible flows, 1997

Properties and observations for Turkel‘s modification

1. Works well for flows which show a global incompressible behavior

2. Works well for inviscid flows

3. For turbulent flow cases often a loss of robustness is observed (convergence problems and NAN, limited CFL number)

4. For flow cases which show both compressible and incompressible effects an improvement of accuracy is questionable

Is there an alternative? Yes, idea of Rossow. (Based on same principal ideas)

When one knows that low Mach preconditioning is a modification of dissipative terms only, one can directly manipulate the weighting operator.

Representation of weigthing with respect to Mach number

$$\int_{\partial\Omega_i} (\mathrm{f_c} \bullet \mathrm{n})ds \approx \sum_{j\in N(i)} \frac{1}{2}\left((\mathrm{f_c} \bullet \mathrm{n})(\mathrm{W}_i) + (\mathrm{f_c} \bullet \mathrm{n})(\mathrm{W}_j) - D_{ij}(\mathrm{W})\right)$$

$$D_{ij}(\mathrm{W}) = \left|\mathrm{A}_{ij}\right|(\Delta \mathrm{W}), \quad \left|\mathrm{A}_{ij}\right| = \mathrm{Q}\left|\Lambda\right|\mathrm{Q}^{-1}$$

$$\begin{aligned}\left|\mathrm{A}\right|_{\mathrm{ef}} = & \frac{1}{2}\frac{\partial \mathrm{W}}{\partial \mathrm{W}_{\mathrm{prim}}}\mathrm{Q}_{\mathrm{prim}}\left[\Lambda + \left|\Lambda\right|_{\mathrm{ef}}\right]\mathrm{Q}_{\mathrm{prim}}^{-1}\frac{\partial \mathrm{W}_{\mathrm{prim}}}{\partial \mathrm{W}} \\ & -\frac{1}{2}\frac{\partial \mathrm{W}}{\partial \mathrm{W}_{\mathrm{prim}}}\mathrm{Q}_{\mathrm{prim}}\left[\Lambda - \left|\Lambda\right|_{\mathrm{ef}}\right]\mathrm{Q}_{\mathrm{prim}}^{-1}\frac{\partial \mathrm{W}_{\mathrm{prim}}}{\partial \mathrm{W}}\end{aligned}$$

Evaluate expression explicitly to get a representation with respect to the Mach number

$$\left|\Lambda\right|_{\mathrm{ef}} = \mathrm{diag}\left(\left|\mathrm{V}\right|_{\mathrm{ef}}, \left|\mathrm{V}\right|_{\mathrm{ef}}, \left|\mathrm{V}\right|_{\mathrm{ef}}, \left|\mathrm{V} + a\mathrm{A}\right|_{\mathrm{ef}}, \left|\mathrm{V} - a\mathrm{A}\right|_{\mathrm{ef}}\right)$$

$$\mathrm{W}_{\mathrm{prim}} := (\rho, u_1, u_2, u_3, p)$$

Care about entropy fix

→ **The correct treatment of the entropy fix while doing the equivalent transformations is of major importance**

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Explicit example of some entries

$$\left(\left|A\right|_{\mathrm{ef}}\right)_{1,1} = \left|V\right|_{\mathrm{ef}} - V M_0^{(1)} + \frac{A}{a}\frac{\|u\|_2^2}{2}(\gamma-1)M_0^{(2)}$$

$$\left(\left|A\right|_{\mathrm{ef}}\right)_{i+1,j+1} = n_j u_i M_0^{(1)} + \delta_{i,j}\left|V\right|_{\mathrm{ef}} + \frac{n_i n_j}{A} a M_0^{(2)}$$

$$+(1-\gamma)u_j\left(\frac{A}{a}u_i M_0^{(2)} + n_i M_0^{(1)}\right) \quad i,j = 1,2,3$$

Representation with respect to speed of sound and Mach number

....

$$M_0^{(1)} := \frac{1}{2}\left(\left|Ma+1\right|_{\mathrm{ef}} - \left|Ma-1\right|_{\mathrm{ef}}\right)$$

$$M_0^{(2)} := \frac{1}{2}\left(-2\left|Ma\right|_{\mathrm{ef}} + \left|Ma+1\right|_{\mathrm{ef}} - \left|Ma-1\right|_{\mathrm{ef}}\right)$$

Terms including Mach number, correct implementation of entropy fix

Explicit example of some entries

$$\left(|\mathrm{A}|_{\mathrm{ef}}\right)_{1,1} = |\mathrm{V}|_{\mathrm{ef}} - \mathrm{V}\mathrm{M}_0^{(1)} + \frac{\mathrm{A}}{a}\frac{\|\mathrm{u}\|_2^2}{2}(\gamma - 1)\mathrm{M}_0^{(2)}$$

$$\left(|\mathrm{A}|_{\mathrm{ef}}\right)_{i+1,j+1} = n_j u_i \mathrm{M}_0^{(1)} + \delta_{i,j}|\mathrm{V}|_{\mathrm{ef}} + \frac{n_i n_j}{\mathrm{A}} a \mathrm{M}_0^{(2)} + (1-\gamma)u_j\left(\frac{\mathrm{A}}{a}u_i\mathrm{M}_0^{(2)} + n_i\mathrm{M}_0^{(1)}\right)$$

Representation with respect to speed of sound and Mach number

....

Idea of Rossow: Replace speed of sound by artificial speed of sound given in the way the Mach number is modified in Turkel's preconditioner

$$\mathrm{M}_r := \min\left\{\max\left\{\mathrm{Ma}^2, K\mathrm{Ma}_\infty^2\right\}, 1\right\}, \qquad K \geq 1$$

$$a_{\mathrm{art}} := \sqrt{\alpha^2\frac{\mathrm{V}^2}{\mathrm{A}^2} + \mathrm{M}_r a^2}, \qquad \alpha := \frac{1}{2}\left(1 - \mathrm{M}_r\right)$$

Explicit example of some entries

$$\left(\left|\mathrm{A}\right|_{\mathrm{ef}}\right)_{1,1} = \left|\mathrm{V}\right|_{\mathrm{ef}} - \mathrm{V}\mathrm{M}_0^{(1)} + \frac{\mathrm{A}}{a}\frac{\left\|\mathrm{u}\right\|_2^2}{2}(\gamma-1)\mathrm{M}_0^{(2)}$$

$$\left(\left|\mathrm{A}\right|_{\mathrm{ef}}\right)_{i+1,\,j+1} = n_j u_i \mathrm{M}_0^{(1)} + \delta_{i,j}\left|\mathrm{V}\right|_{\mathrm{ef}} + \frac{n_i n_j}{\mathrm{A}} a \mathrm{M}_0^{(2)}$$

$$+ (1-\gamma) u_j \left(\frac{\mathrm{A}}{a} u_i \mathrm{M}_0^{(2)} + n_i \mathrm{M}_0^{(1)} \right)$$

Representation with respect to speed of sound and Mach number

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Idea of Rossow: Replace speed of sound by artificial speed of sound given in the way the Mach number is modified in Turkel's preconditioner

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$$a_{\mathrm{art}} := \sqrt{\alpha^2 \frac{\mathrm{V}^2}{\mathrm{A}^2} + \mathrm{M}_r a^2}, \qquad \alpha := \frac{1}{2}(1-\mathrm{M}_r)$$

Description of technique in several papers of Rossow, Rossow & Swanson

→ Did not work at all in all my tries for unstructured codes

Explicit example of some entries

Rossow/Swanson modification

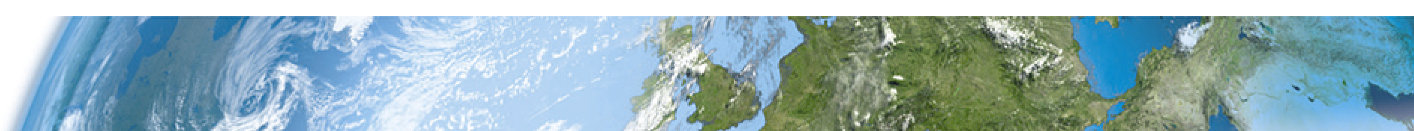
$$\mathrm{M}_r := \min\left\{ \max\left\{ \mathrm{Ma}^2, K\mathrm{Ma}_\infty^2 \right\},\ 1 \right\}, \qquad K \geq 1$$

$$a_{\mathrm{art}} := \sqrt{\alpha^2 \frac{\mathrm{V}^2}{\mathrm{A}^2} + \mathrm{M}_r a^2}, \qquad \alpha := \frac{1}{2}\left(1 - \mathrm{M}_r\right)$$

Additional modification required:

$$\left|\mathrm{V}\right|_{\mathrm{ef,\,art}} := \max\left\{ \left|\mathrm{V}\right|, \omega\left(\left|\mathrm{V}\right| + a_{\mathrm{art}} \mathrm{A} \right) \right\}, \qquad \omega \in \left[\frac{1}{5},\ 1 \right]$$

→ Modification only in momentum equations



Remarks

1. No clue, why the additional (non reported) modification is required

2. Like the parameter K the further parameter ω is undesired

3. The modified matrix and its properties are not straightforward to understand
   - So far I was only able to compute two of its eigenvalues and one of its eigenvectors (computer algebra systems did not yield any success) → further analysis required

4. Asymptotic behavior still needs to be determined

5. In the literature the low Mach modifications are in most of the results applied to globally incompressible flows, and only rarely one has results with respect to e.g. high-lift configurations → Close this gap

Solution algorithm:
FAS multigrid + Preconditioned Runge Kutta smoother

$$W^{(0)} = W^{(n)}$$

$$W^{(1)} = W^{(0)} - CFL\,\alpha_{21}\Delta t\, P^{-1} R(W^{(0)})$$

$$W^{(2)} = W^{(1)} - CFL\,\alpha_{32}\Delta t\, P^{-1} R\,(W^{(1)})$$

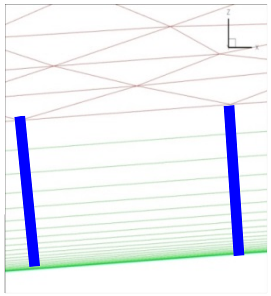
$$W^{(3)} = W^{(0)} - CFL\,\alpha_{43}\Delta t\, P^{-1} R\,(W^{(2)})$$

$$W^{(n+1)} = W^{(3)}$$

$$\alpha_{21} = 0.15,\quad \alpha_{32} = 0.4,\quad \alpha_{43} = 1$$

Preconditioner

$$\boxed{P^{-1} = \left(\frac{1}{CFL\Delta t} I + \frac{\partial R^{app}}{\partial W} \right)^{-1,app}}$$



Preconditioner	2nd order	1st order
Linear solution algorithms	GmRes via Finite differences, Preconditioned by 1st order + symm. Line Gauss-Seidel	Symm. line Gauss-Seidel (5 sweeps)

DLR

Solution algorithm:
FAS multigrid + Preconditioned Runge Kutta smoother

$$W^{(0)} = W^{(n)}$$

$$W^{(1)} = W^{(0)} - CFL\,\alpha_{21}\Delta t\, P^{-1} R(W^{(0)})$$

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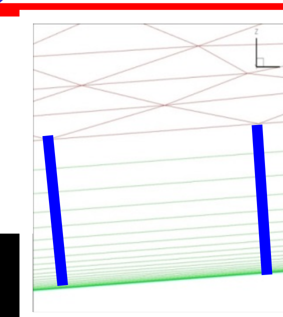
Preconditioner

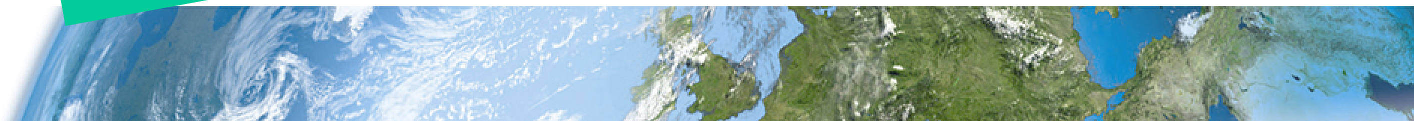
$$\boxed{P^{-1} = \left(\frac{1}{CFL\Delta t} I + \frac{\partial R^{\mathrm{app}}}{\partial W}\right)^{-1,\mathrm{app}}}$$



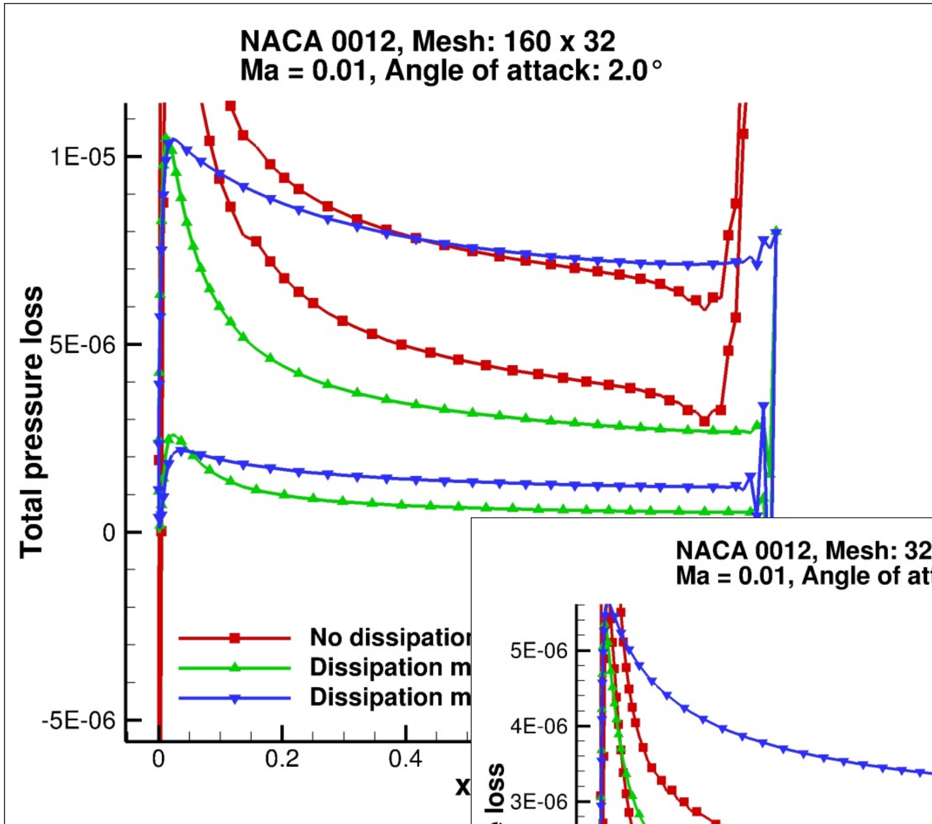
Preconditioner	2nd order	1st order
Linear solution algorithms	G…	Symm. line Gauss-S… (5 …

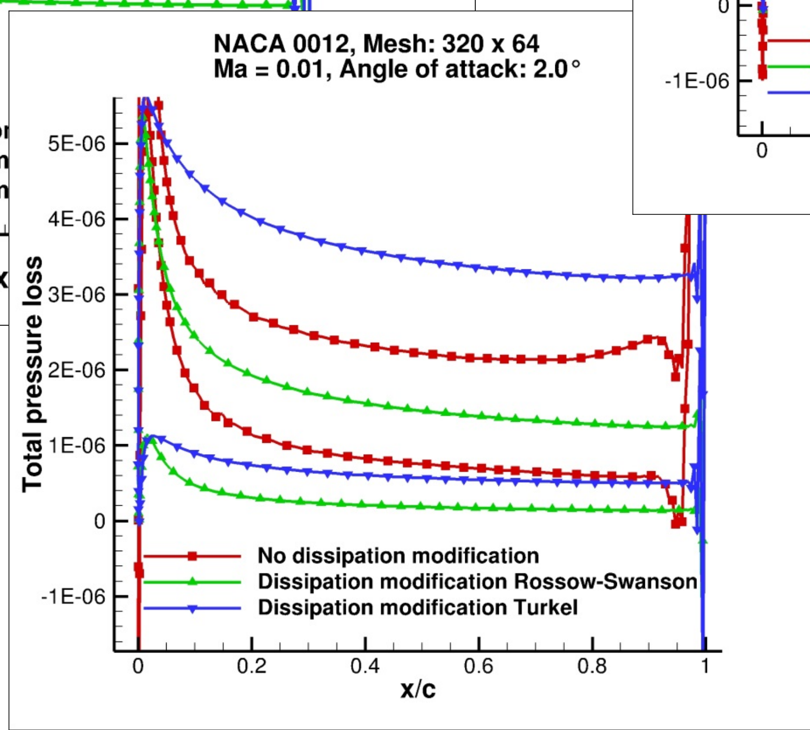
Used for complex 3d high-lift test cases
→ Computational expensive

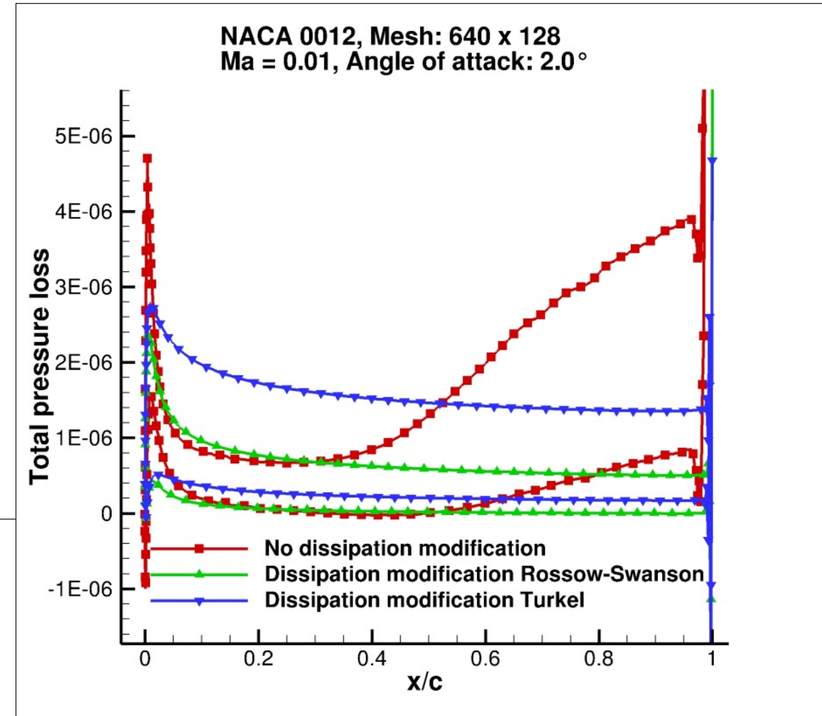
Used for other test cases



Example: Inviscid flow over NACA0012, Ma = 0.01

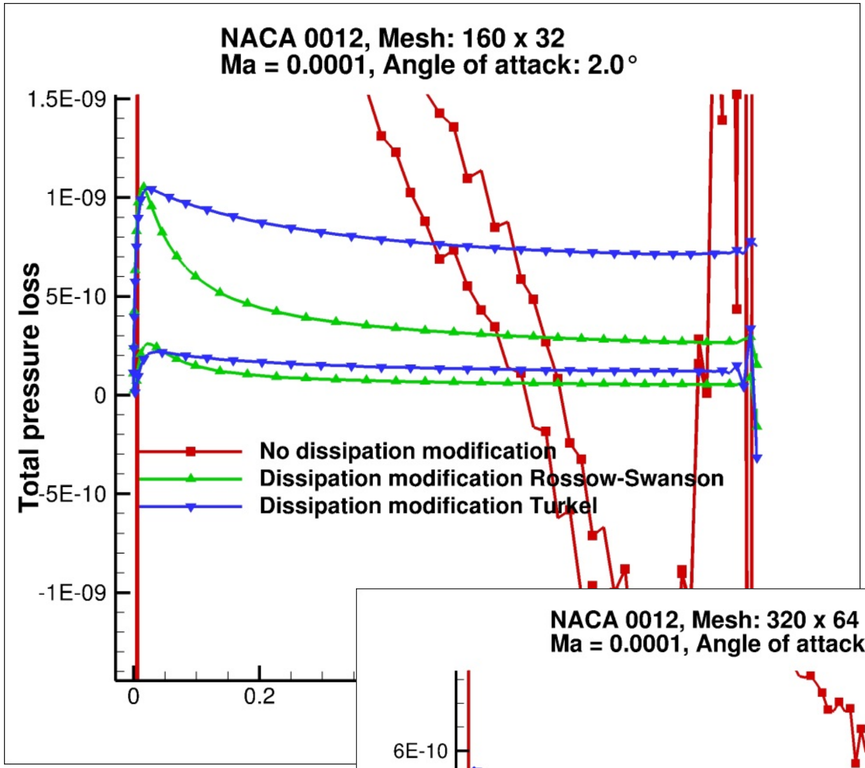


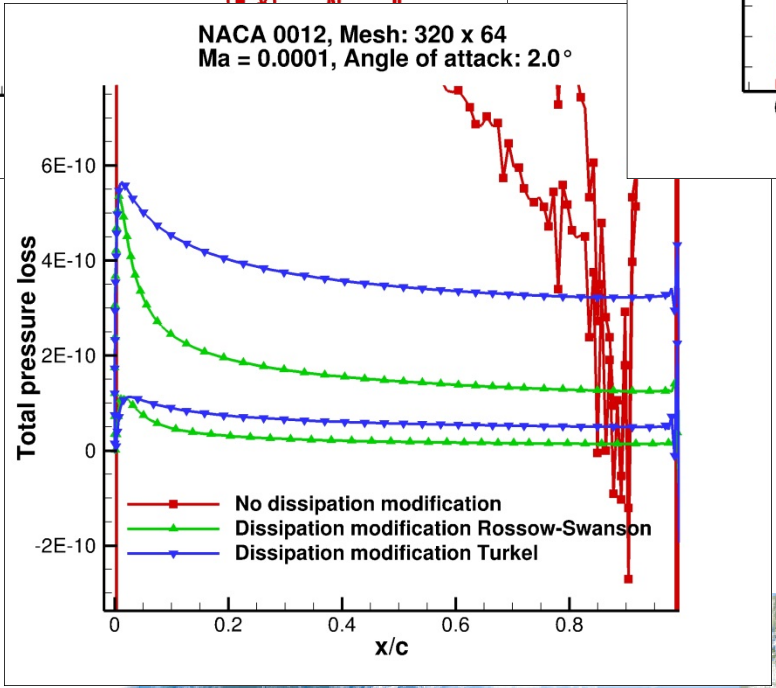


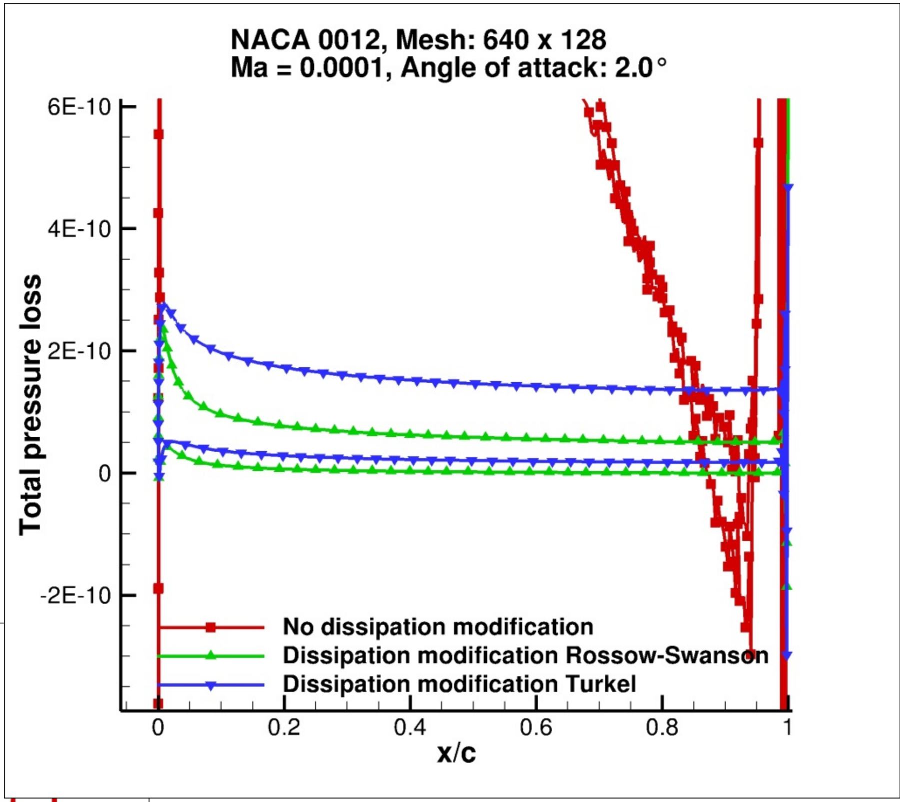


Rossow modification shows most accurate results

Example: Inviscid flow over NACA0012, Ma = 0.0001

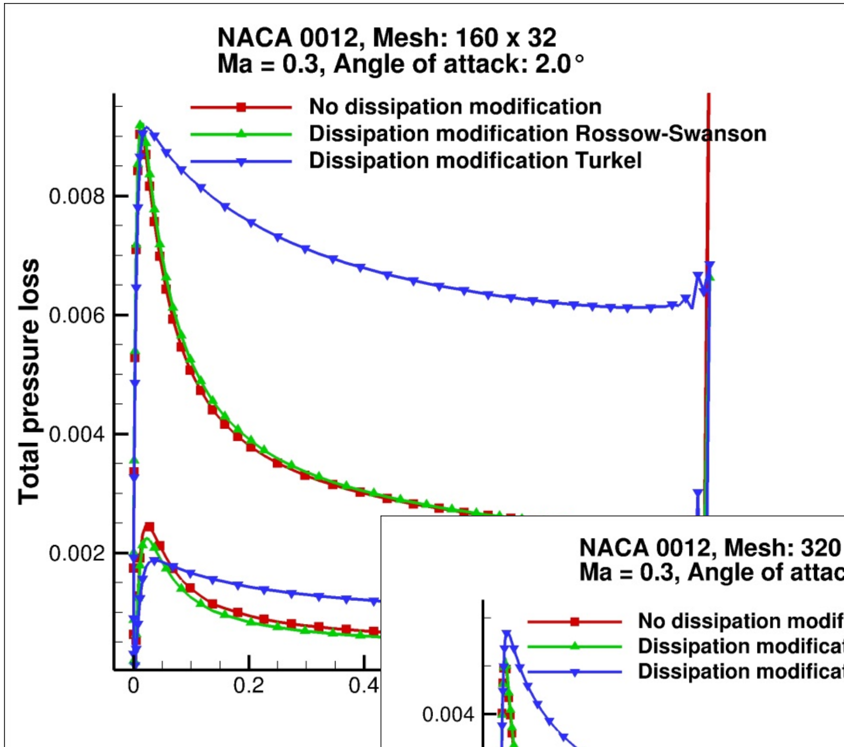


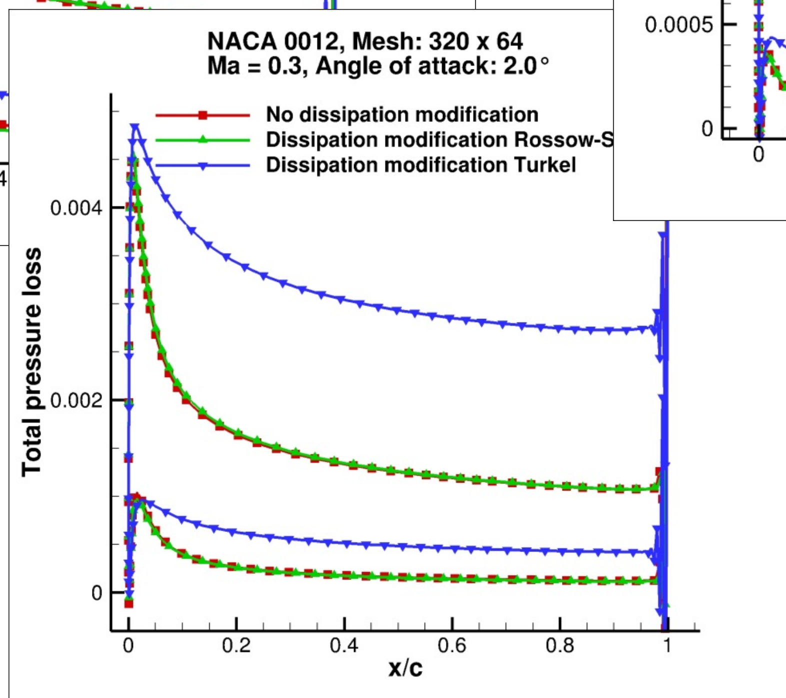


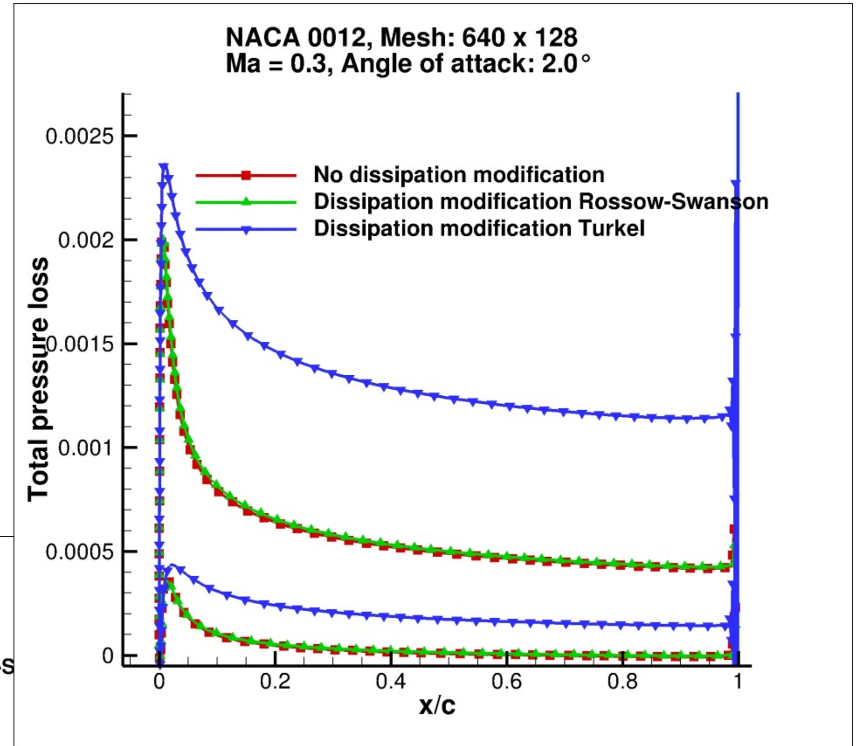


Rossow modification shows most accurate results

Example: Inviscid flow over NACA0012, Ma = 0.3

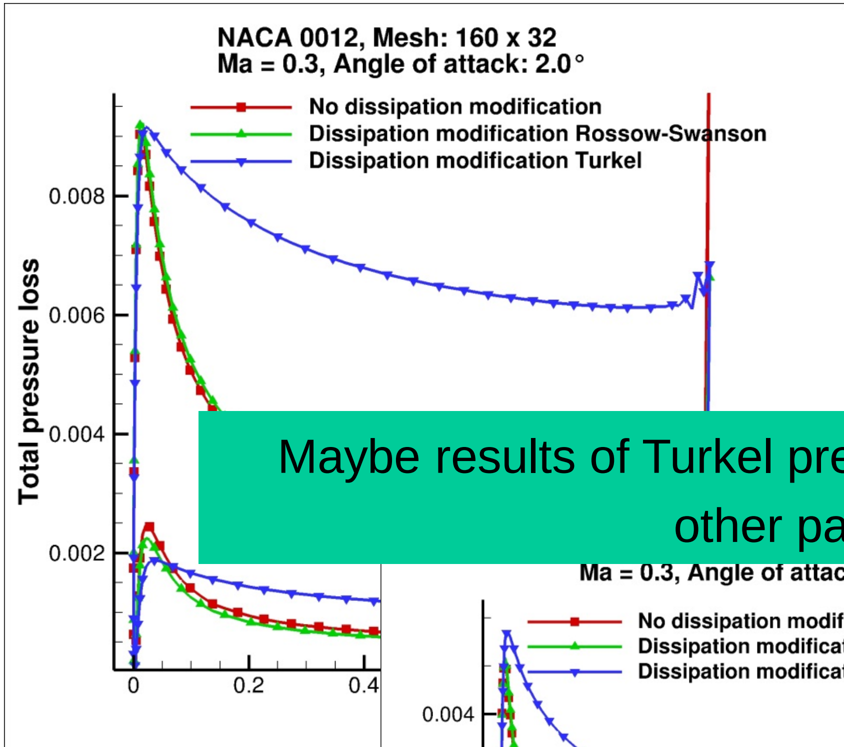


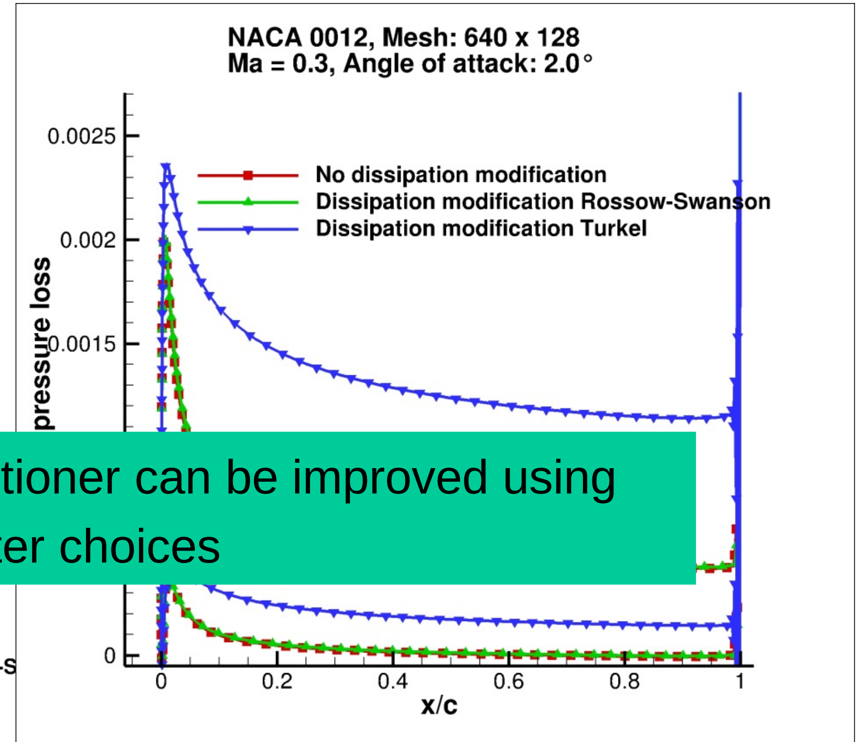


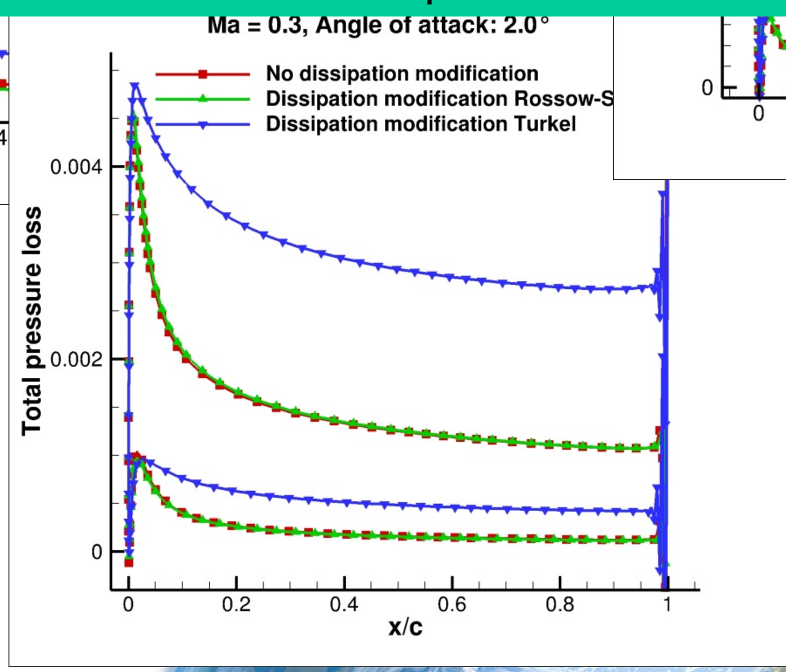


Rossow preconditioner is comparable to non preconditioned case; implementation of Turkel preconditioner shows loss of accuracy

Example: Inviscid flow over NACA0012, Ma = 0.3



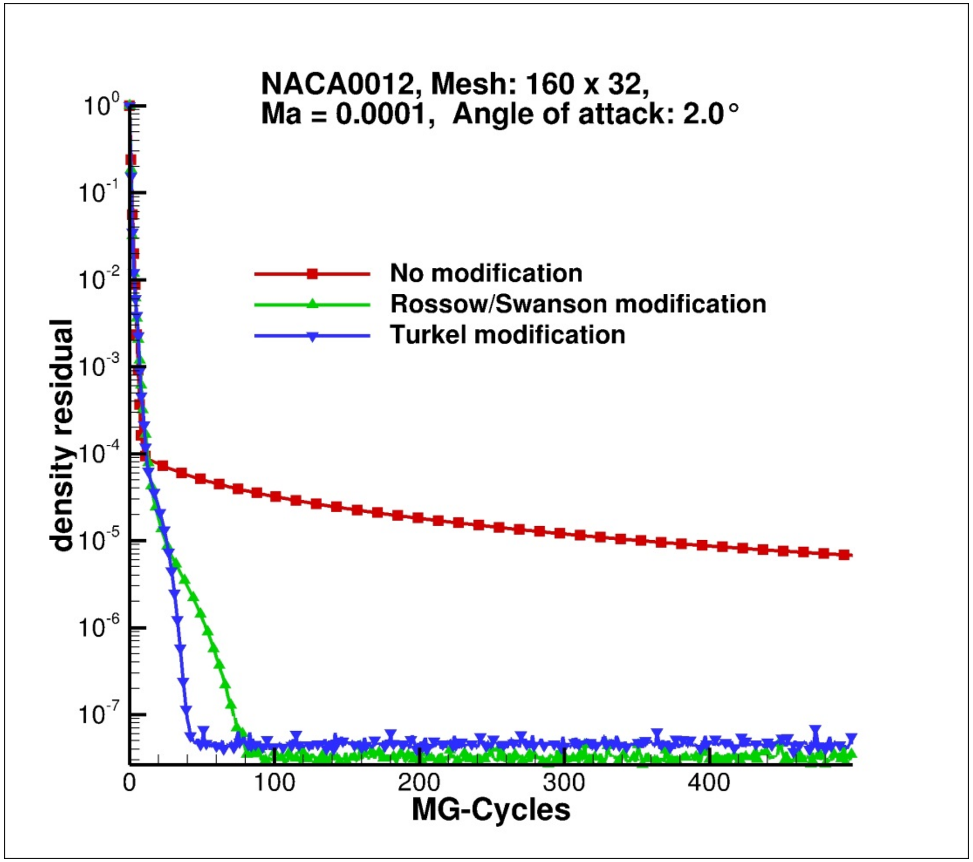


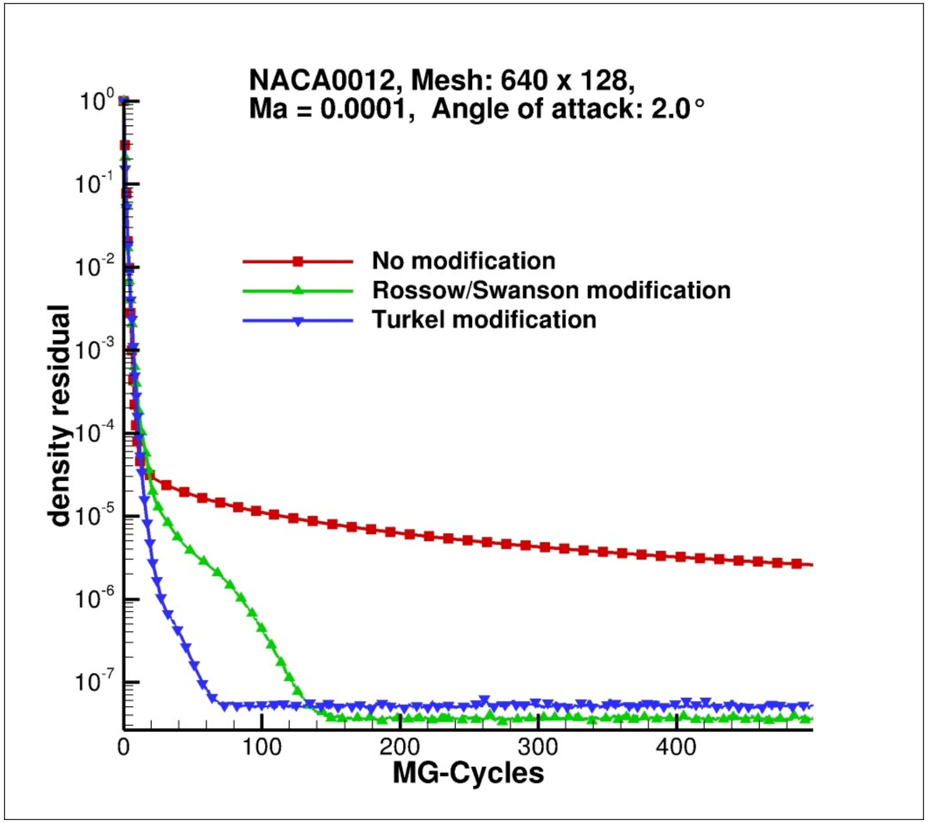


Maybe results of Turkel preconditioner can be improved using other parameter choices

Rossow preconditioner is comparable to non preconditioned case; implementation of Turkel preconditioner shows loss of accuracy

Example: Inviscid flow over NACA0012, Ma = 0.0001



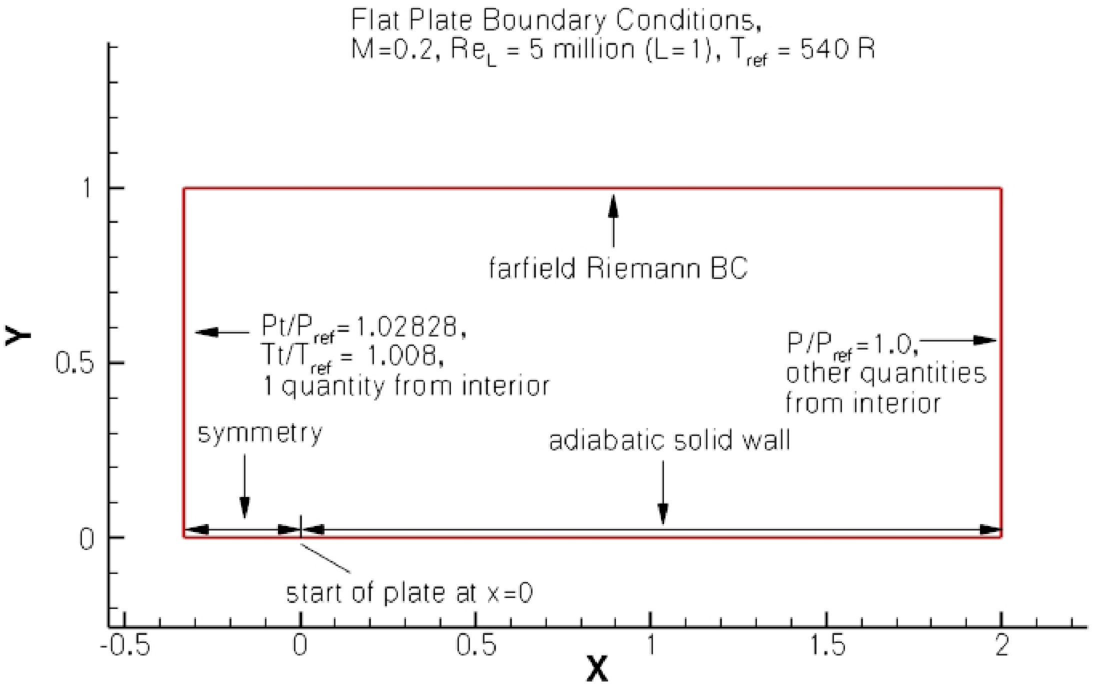


→ For low Mach numbers residuals do not converge to machine accuracy (documented in the literature by several authors, can be fixed using pressure corrections)

DLR

Turbulent flat plate

Use case from NASA Turbulence Modeling Ressource.
(turbmodels.larc.nasa.gov)

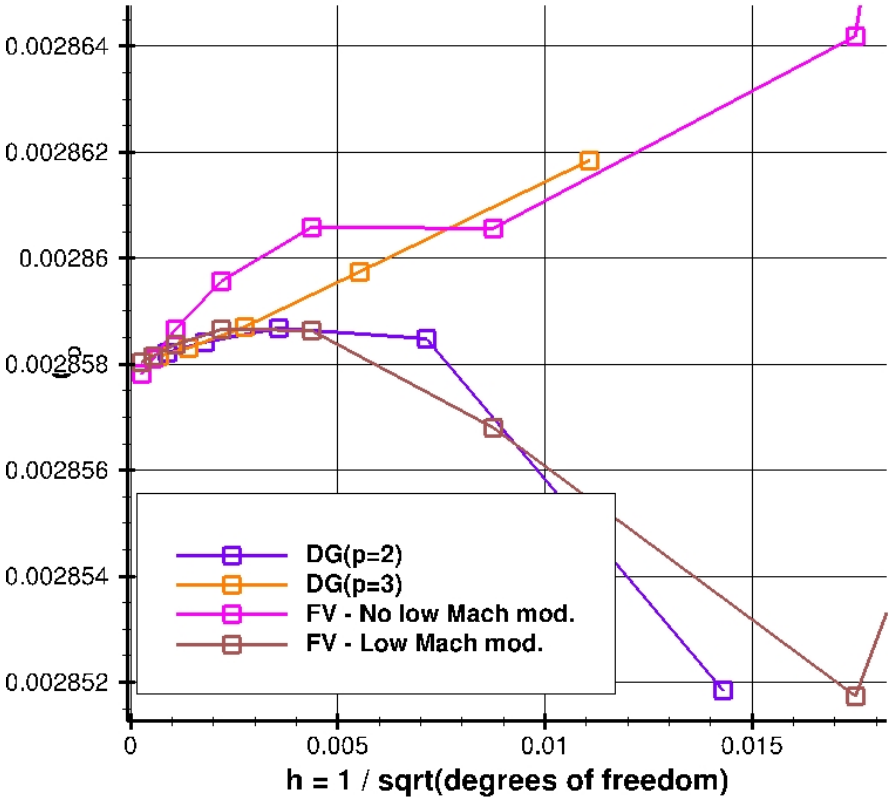


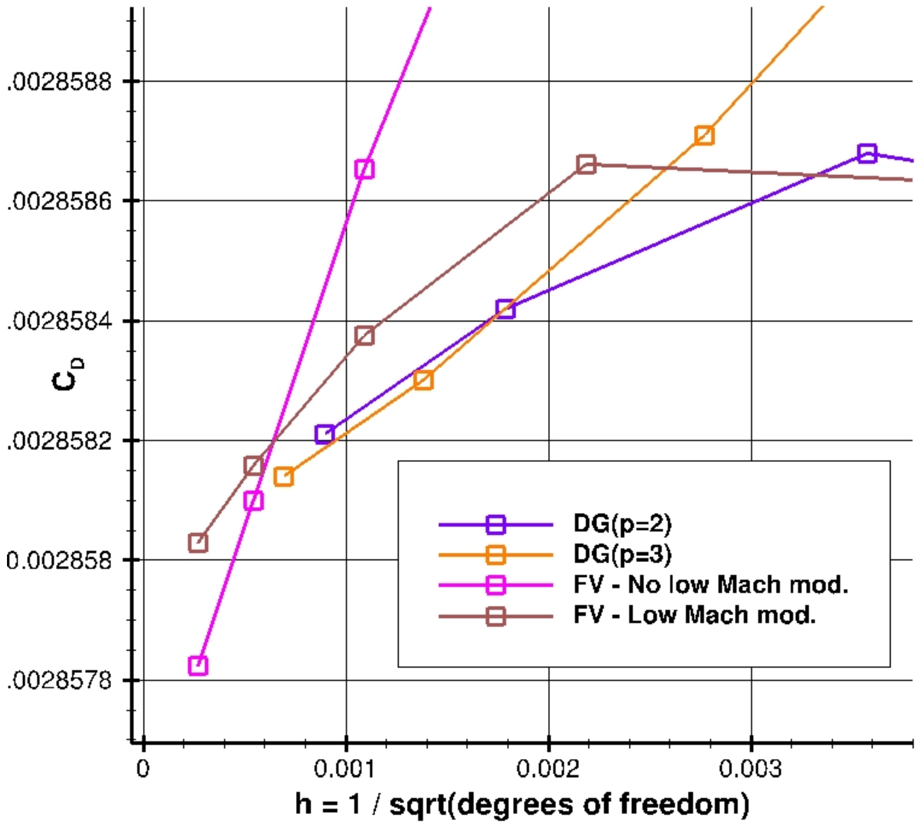
Nested structured meshes

- 35 x 25
- 69 x 49
- 137 x 97
- 273 x 193
- 545 x 385
- 1089 x 769 ~ 0.8e6 NDOF
- 2177 x 1537 ~ 3.3e6 NDOF
- 4353 x 3073 ~ 13.4e6 NDOF

Ma = 0.2, Re = 5.0e6

Turbulent flat plate: Results



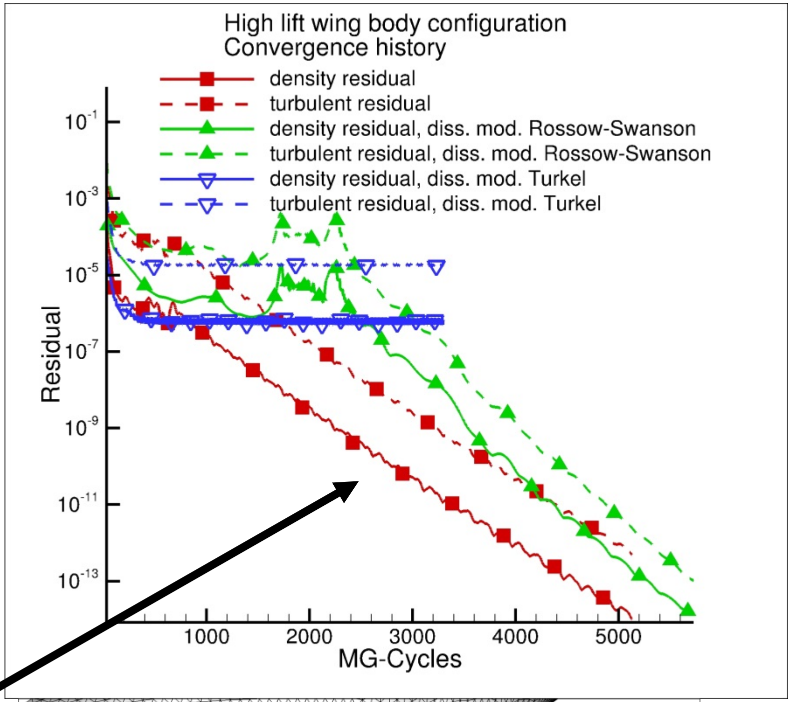


Comparison with DG code of order 3 and order 4

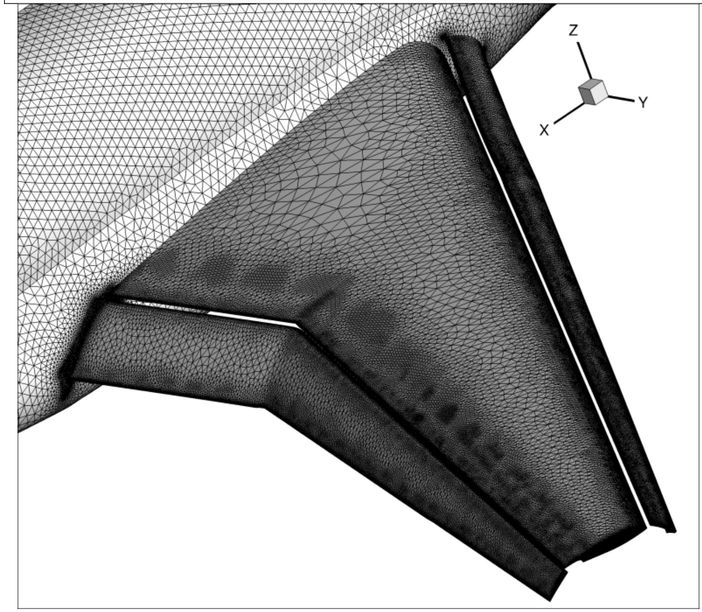
→ Low Mach modified values (only Rossow) are closer to the results of high order code

Numerical example
3D High-lift

- A310 high-lift wing body configuration
- Ma = 0.1816
- α = 24.0°
- Re = 15.495e6
- CENTAUR mesh
    - 10733766 points
    - 11590308 tetras
    - 17128338 prisms
    - 1523 pyramids



Turkel modification does not converge, but Rossow/Swanson modification does

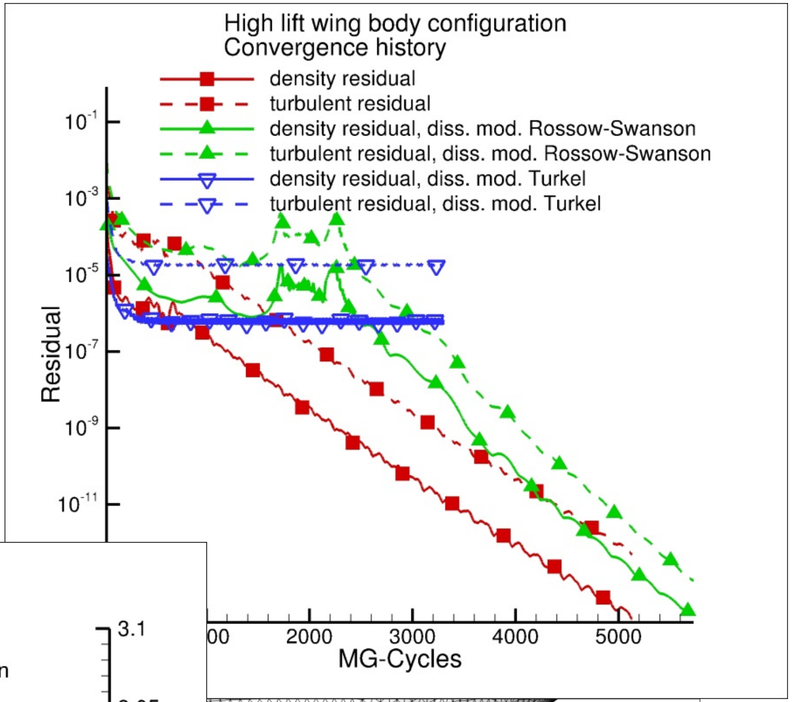


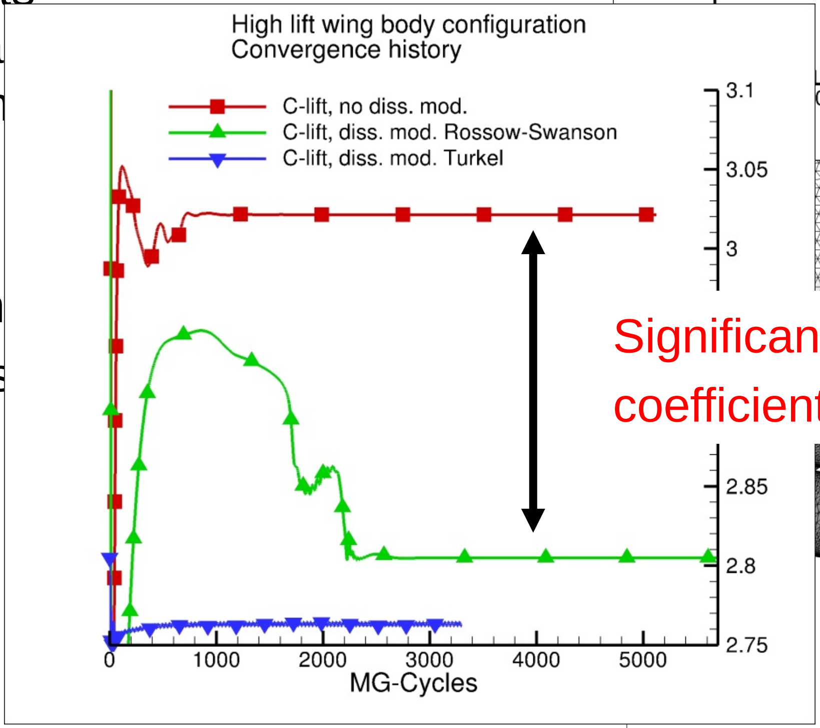
-No. of points: 11e6

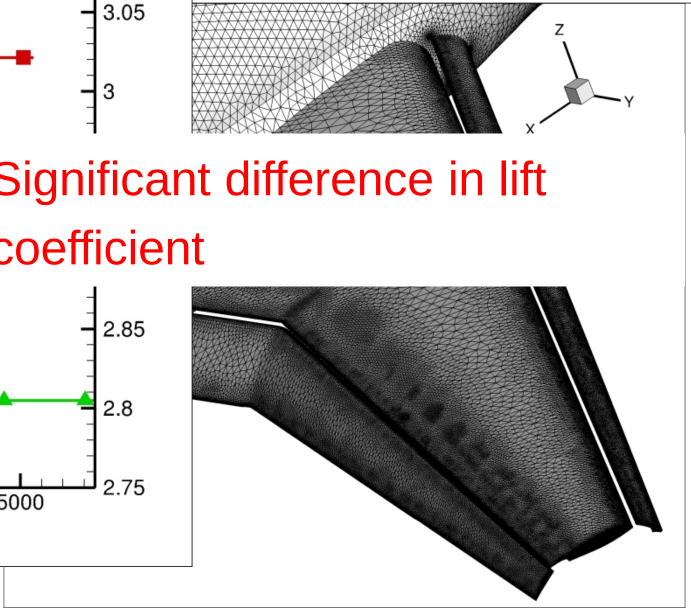
Numerical example
3D High-lift

- A310 high-lift wing body configuration
- Ma = 0.1816
- $\alpha = 24.0^\circ$
- Re = 15.495e6
- CENTAUR mesh
    - 10733766 points
    - 11590308 tetra
    - 17128338 prisn
    - 1523 pyramids

Turkel precondition
converge, but Ross
does

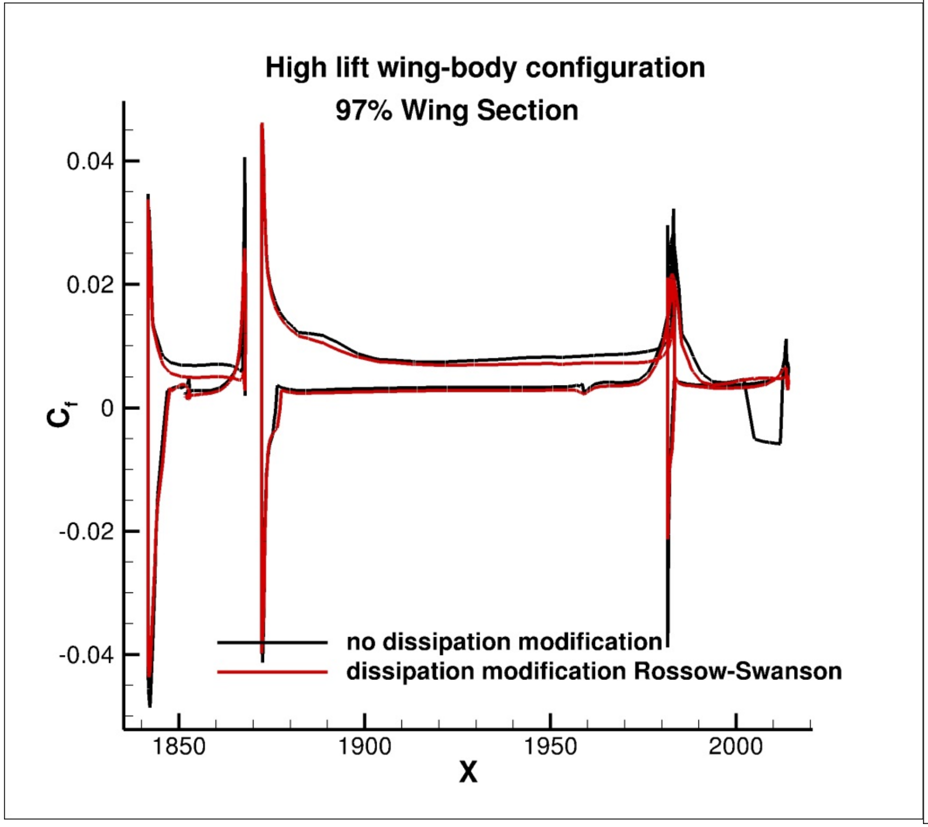


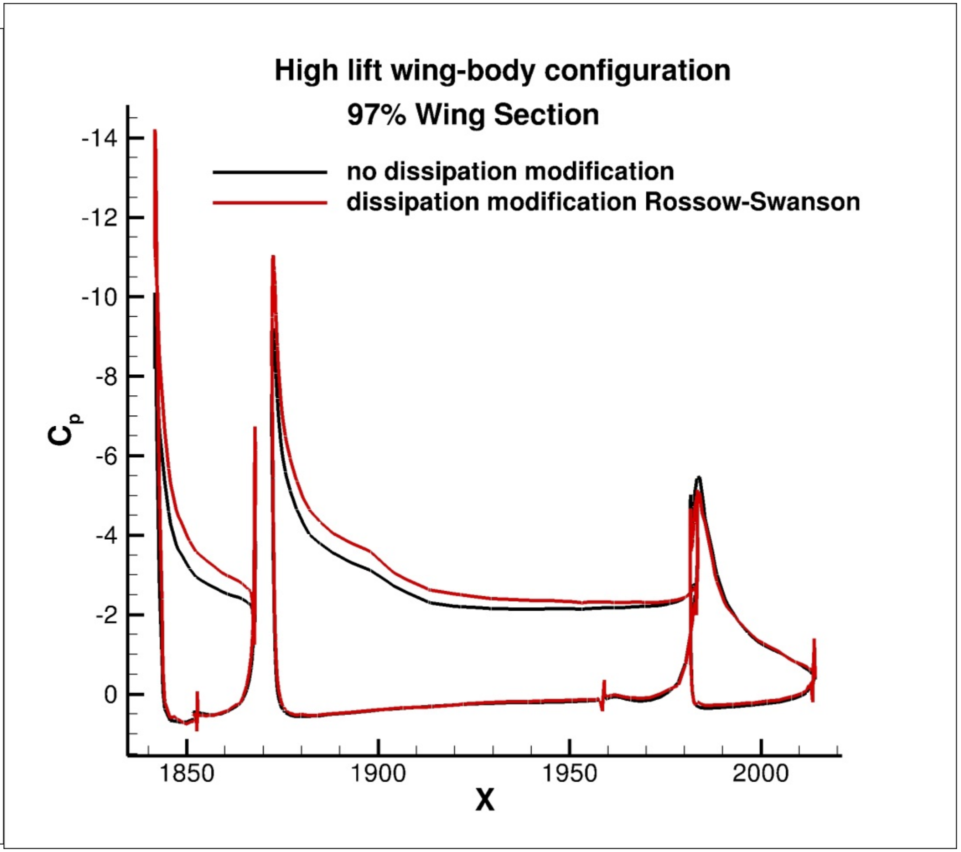




Significant difference in lift coefficient

Numerical example
3D High-lift

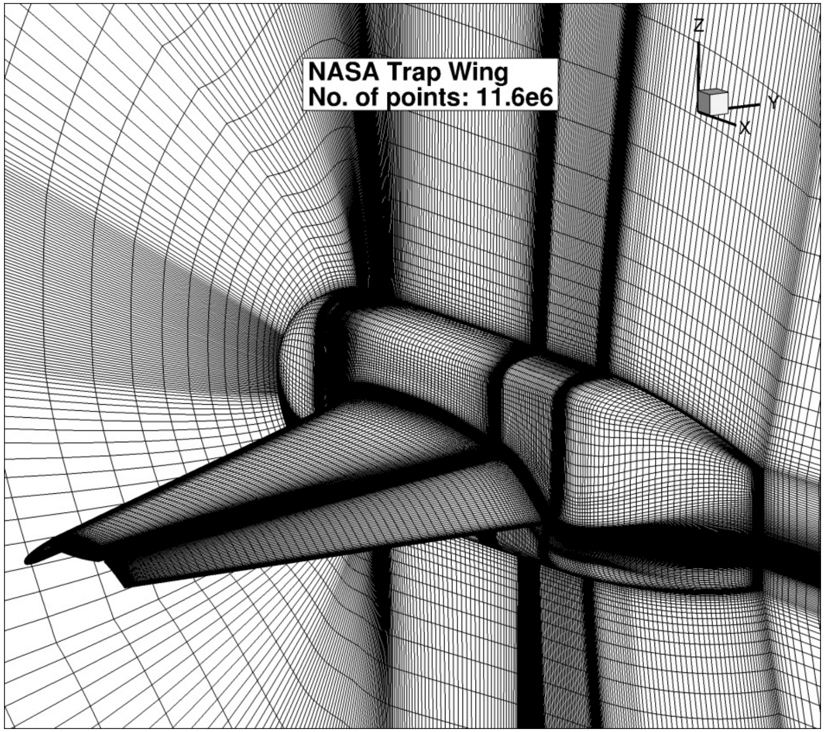




NASA Trap Wing, Ma = 0.2, Re = 4.3e6

Structured Mesh

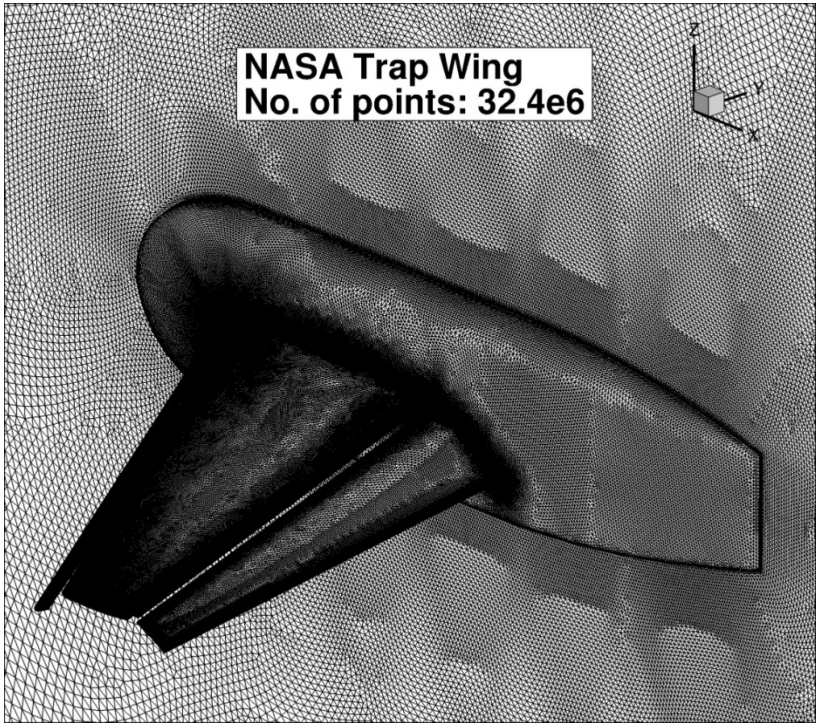
- ~11.6e06 NDOF



Unfortunately no sequence of meshes available and no opportunity to refine given mesh

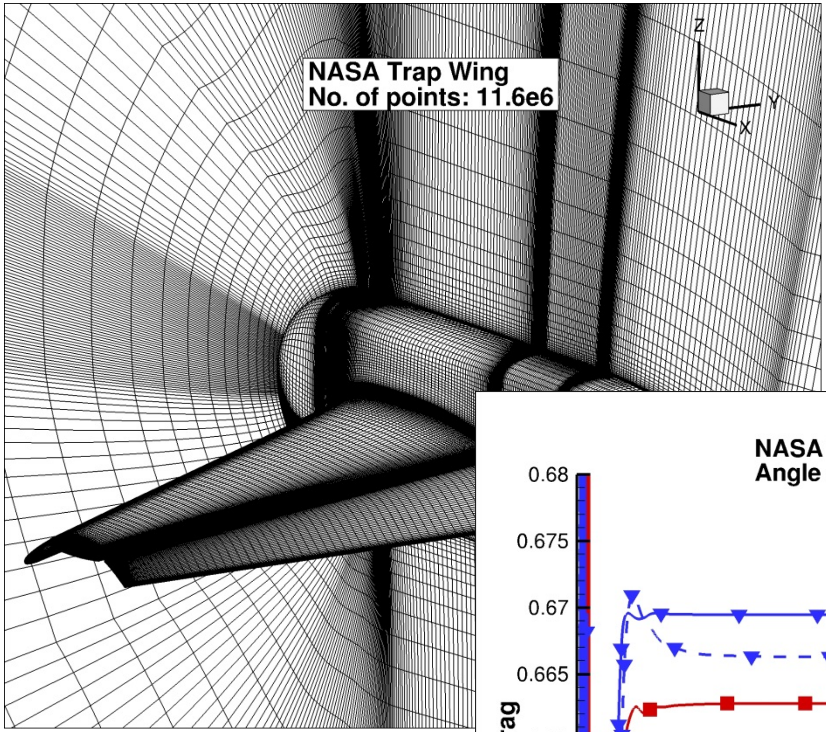
Sequence of unstructured meshes used at the HiLiftPW 1

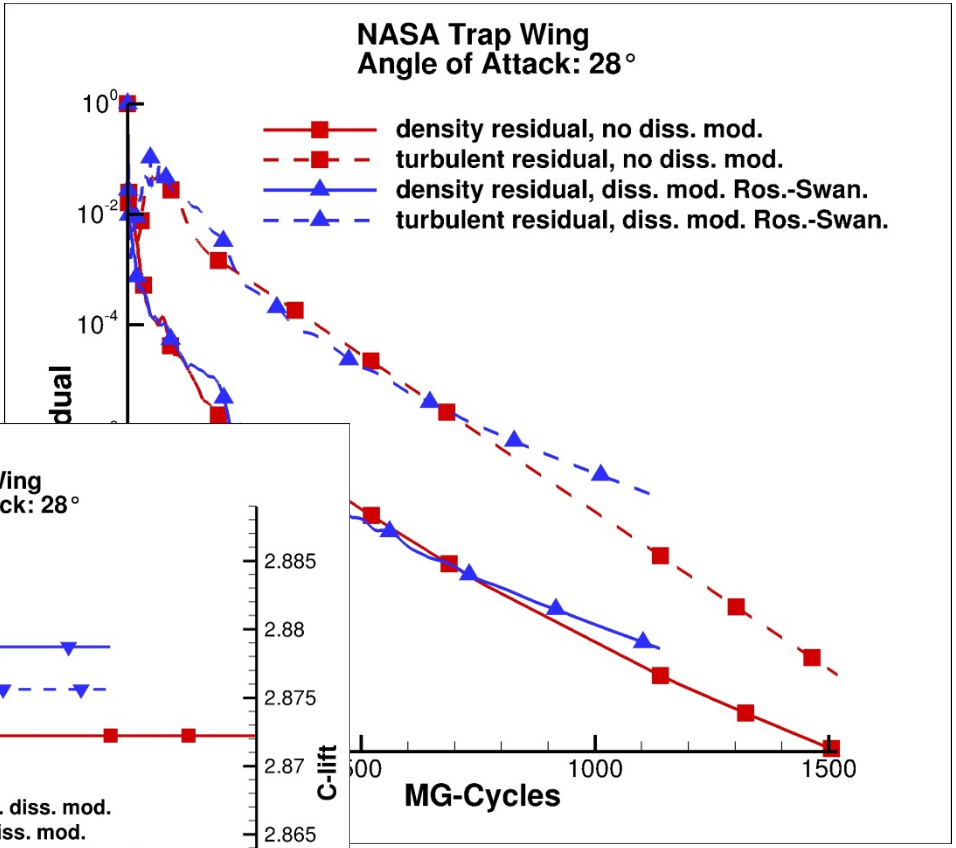
- Coarse Mesh: 3.7e6 NDOF
- Medium Mesh: 11.0e6 NDOF
- Fine Mesh: 32.e6 NDOF

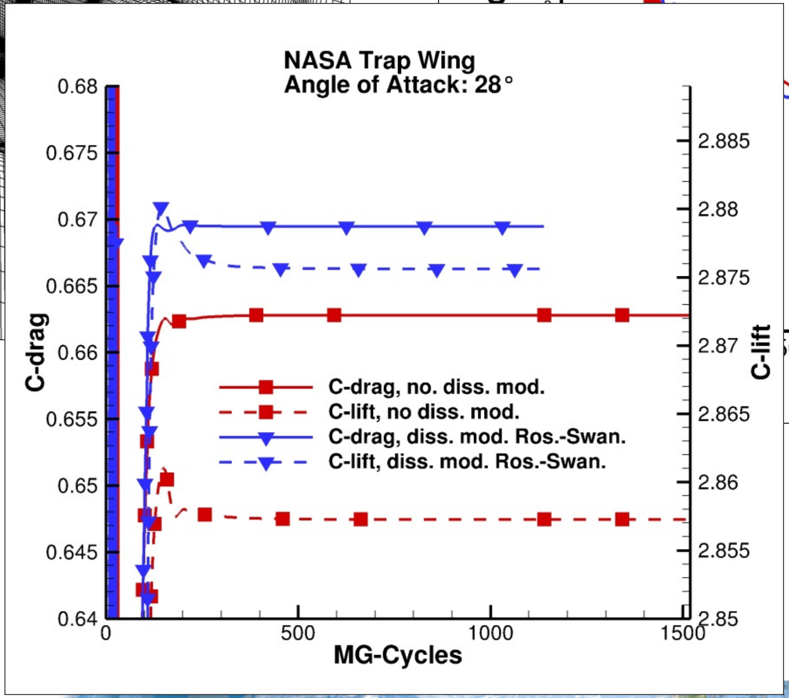


NASA Trap Wing, Ma = 0.2, Re = 4.3e6

Structured mesh results only for AOA = 28°



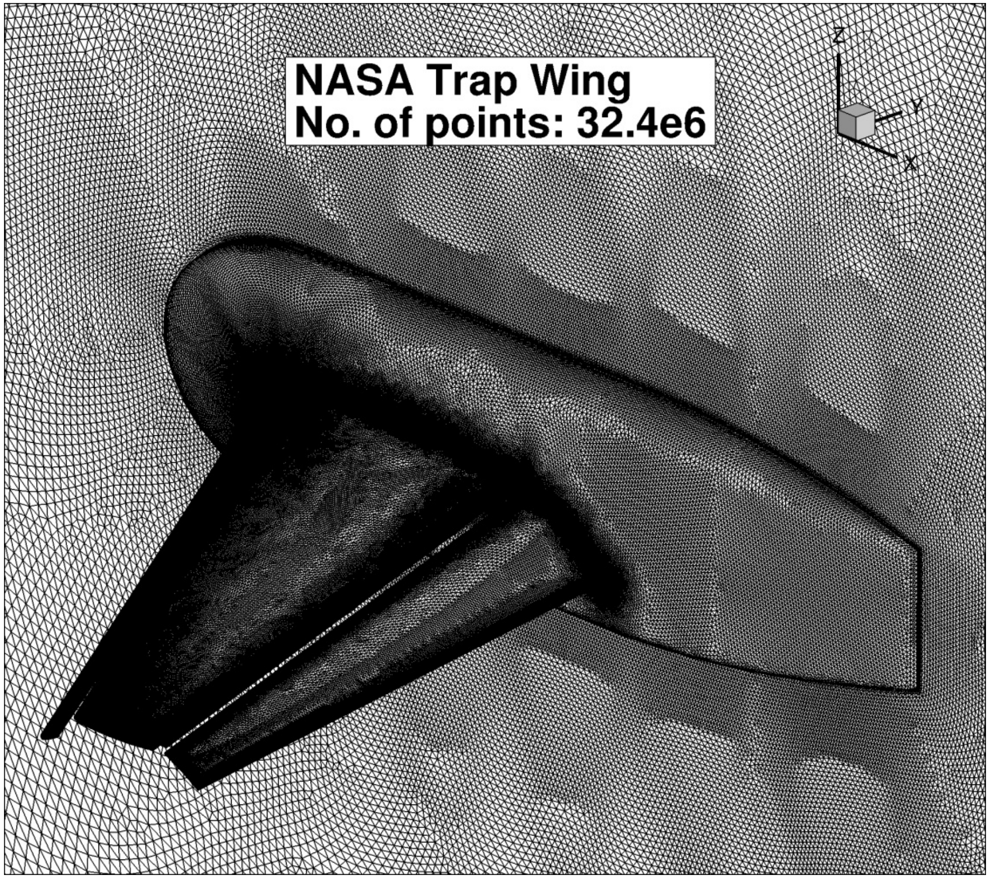


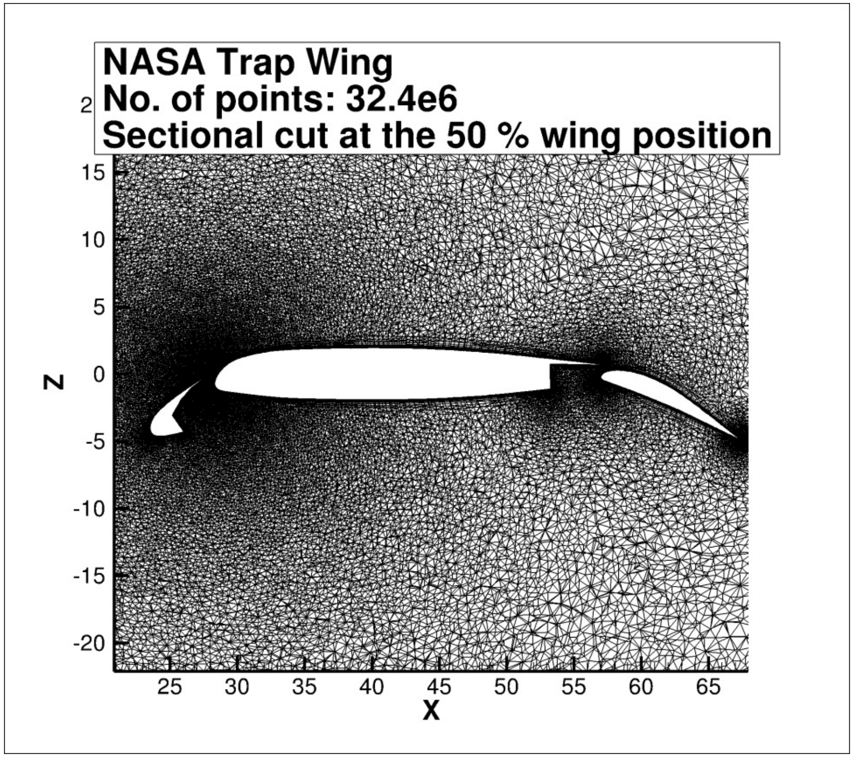


NASA Trap Wing, Ma = 0.2, Re = 4.3e6

Unstructured mesh results for

AOA = 13°, 28°, 32°, 34°, 37°





- Coarse Mesh: 3.7e6 NDOF
- Medium Mesh: 11.0e6 NDOF
- Fine Mesh: 32.ee6 NDOF

VGRID Meshes used at HiLiftPW 1

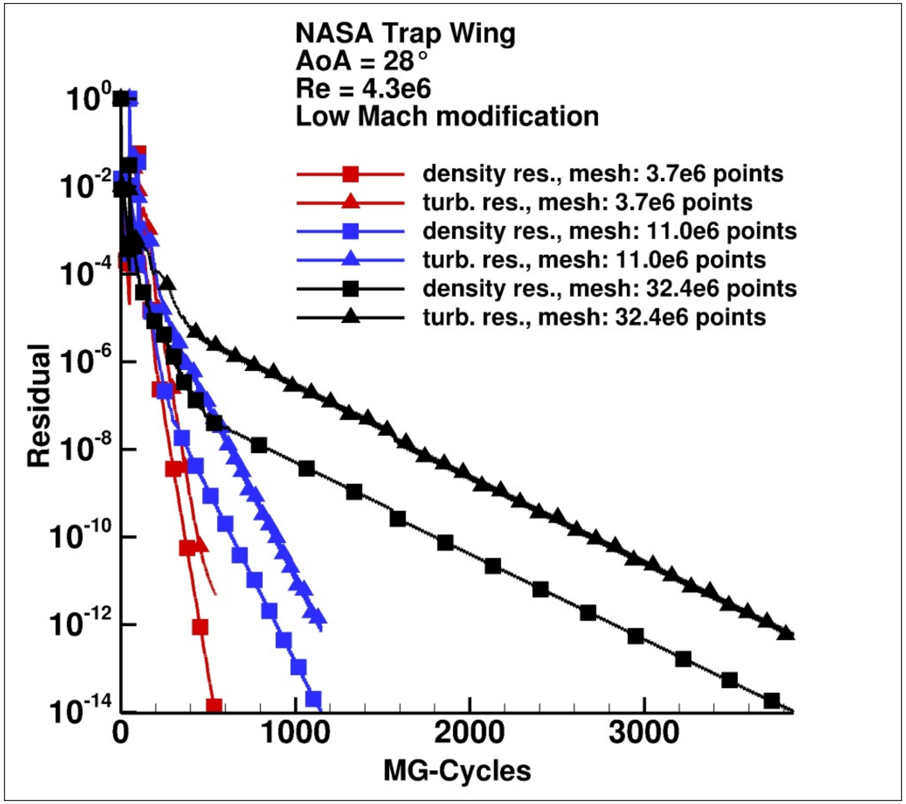
DLR

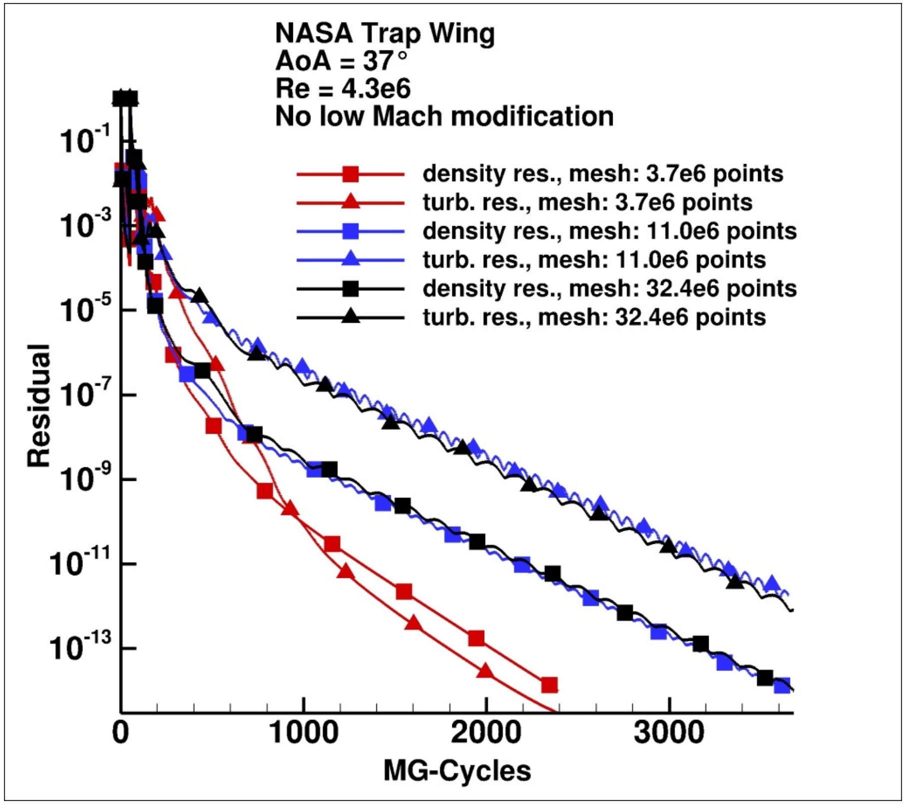
NASA Trap Wing, Ma = 0.2, Re = 4.3e6

Unstructured mesh results for

AOA = 13°, 28°, 32°, 34°, 37°

- **In all experiments the density residual has been reduced by 14 orders of magnitude**



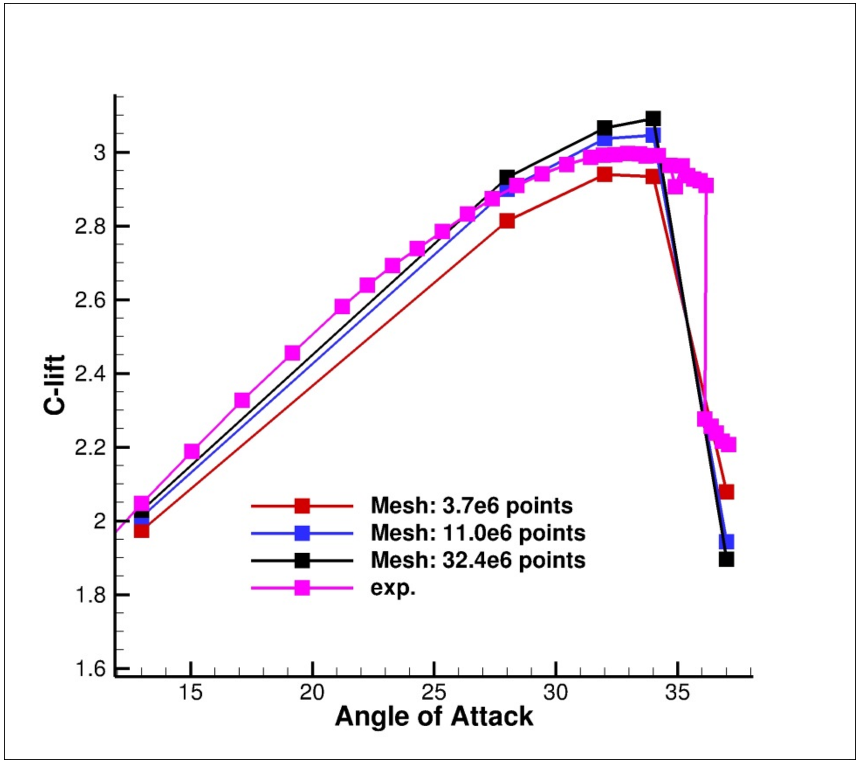


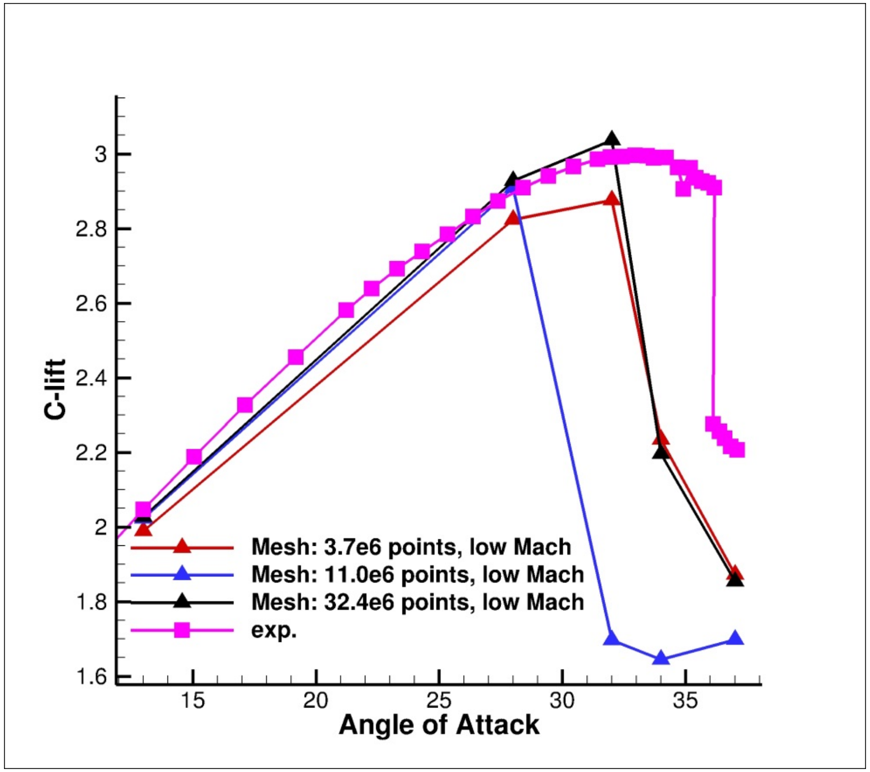
Ex.: AoA = 28°, With Low Mach mod.

Ex.: AoA = 37°, No Low Mach mod.

DLR

NASA Trap Wing, Ma = 0.2, Re = 4.3e6, Lift curve



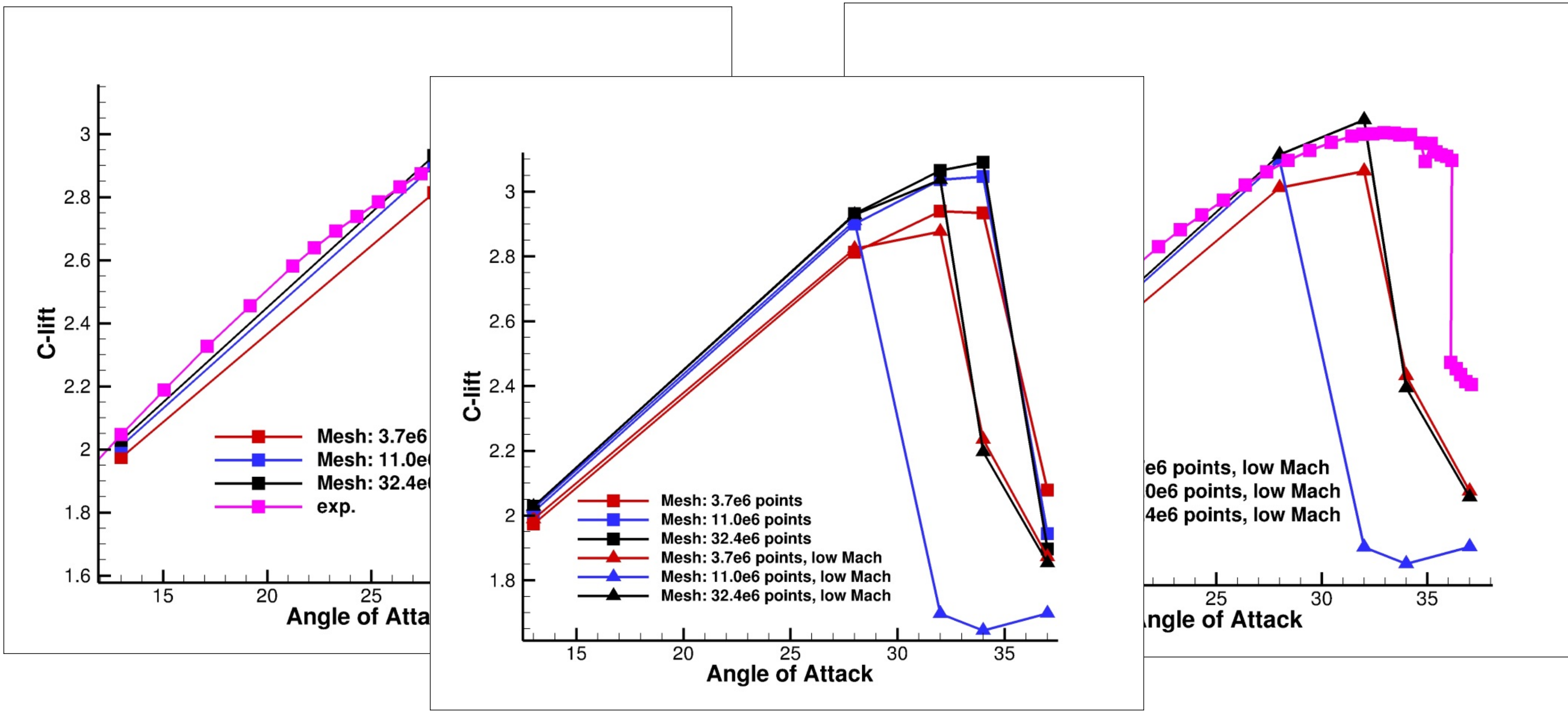


Significant differences in lift coefficients for higher angles of attack for low Mach modified and non low Mach modified computations

→ Missing grid resolution?

DLR

NASA Trap Wing, Ma = 0.2, Re = 4.3e6, Lift curve



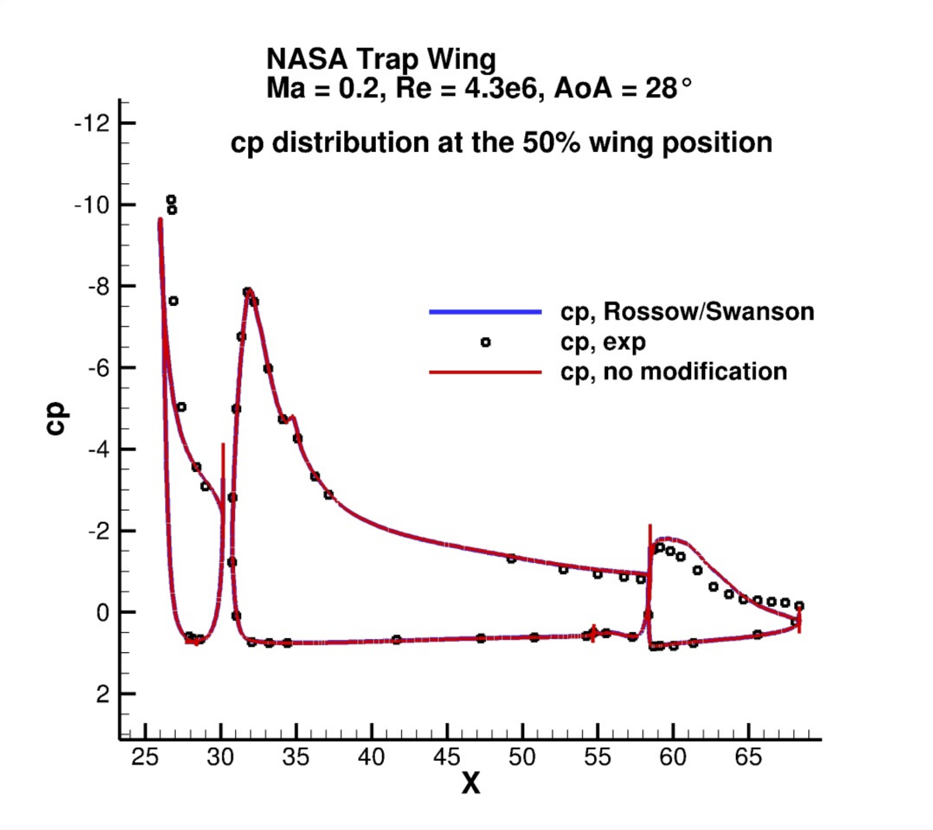
Significant differences in lift coefficients for higher angles of attack for low Mach modified and non low Mach modified computations

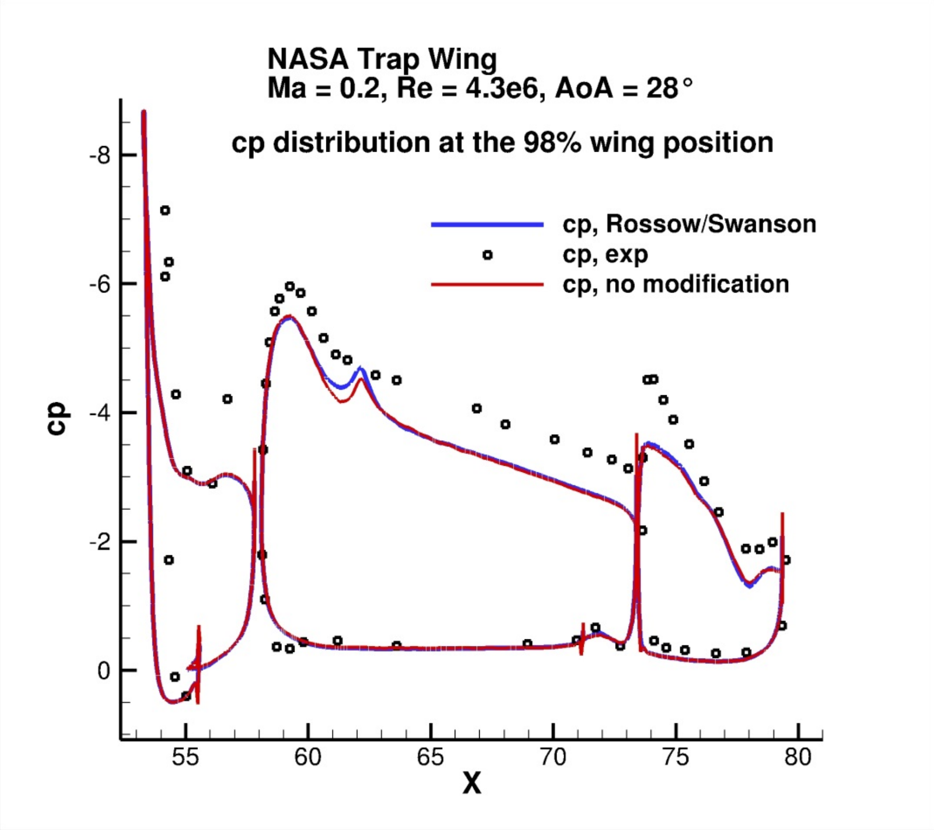
→ Missing grid resolution?

→ Low Mach modified scheme predicts earlier flow separation

DLR

NASA Trap Wing, Ma = 0.2, Re = 4.3e6, AoA = 28°

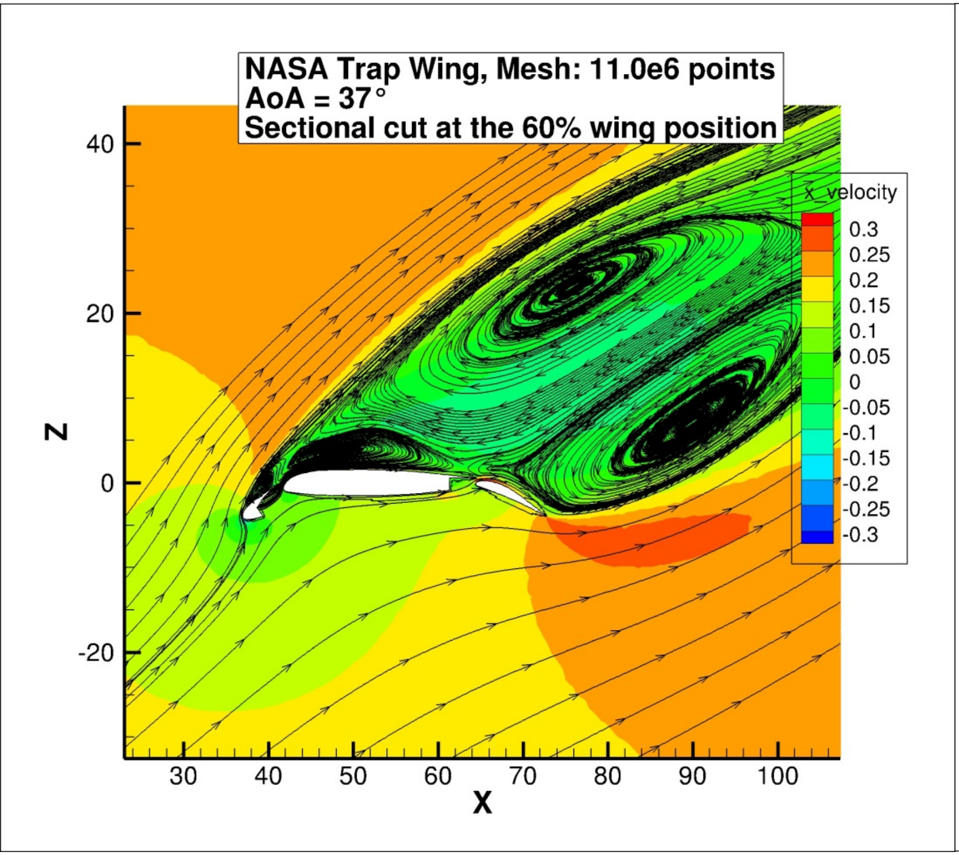


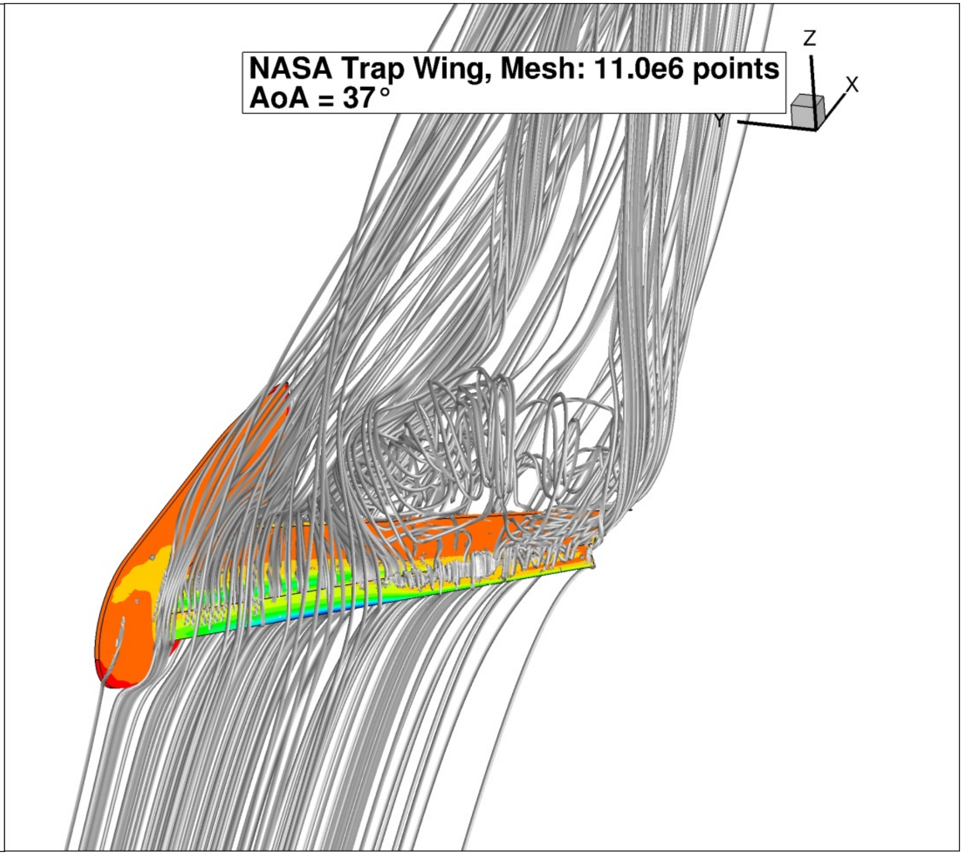


Sectional cut at the 50% wing position

Sectional cut at the 98% wing position

Illustration of separation for the 37° case





Several macroscopic vortices which exhibit a steady state behavior for the RANS equations combined with SA-turbulence model

Conclusion, next steps

1. The low Mach modification of Rossow/Swanson (with further adjustments) has been succesfully integrated into an unstructured node centered finite Volume scheme
2. Rossow / Swanson preconditioner can be implemented in a robust fashion into a RANS code in combination with an implicit solution methodology
3. Several examples of
   a) globally incompressible flow and
   b) incompressible flows with locally strong compressible effects
   c) structured and unstructured meshes

   have demonstrated the superiority of the Rossow/Swanson low Mach modification compared to the Turkel's one w.r.t. robustness
4. The gap is closed to show that low Mach modifications can be succesfully applied to industrial relevant 3d High-Lift test cases
5. Both, the idea of the Turkel and Rossow/Swanson preconditioner can also be applied to other upwinding technqiues (e.g. AUSM, AUSMDV)

Next steps

1. Full eigendecomposition for low Mach modified operator is required
    - Base entropy fix on largest eigenvalue (maybe further improvement)
    - Note, a complete eigendecomposition is not straighforward to do

2. Compare results with results obtained by an incompressible code

3. Clarify accuracy issues for incompressibel flows with locally compressible flow effects (e.g. high-lift)

4. **Further grid refinement studies to better understand difference between non low Mach modified results and low Mach modified results**

5. **Analysis to better understand the differences of Rossow‘s/Swanson‘s and Turkel‘s modification**

Thank you!

Questions?

Low Mach modification (Turkel)

$$\frac{d}{dt}\int_{\Omega} \mathrm{W} dx + \int_{\partial\Omega} \left(\mathrm{f_c} \bullet \mathrm{n} - \mathrm{f_v} \bullet \mathrm{n}\right) ds = 0$$

$$\frac{d}{dt}\int_{\Omega} \mathrm{W} dx + \int_{\partial\Omega} \mathrm{P_{LM}} \left(\mathrm{f_c} \bullet \mathrm{n} - \mathrm{f_v} \bullet \mathrm{n}\right) ds = 0$$

Low Mach modification

$$\int_{\partial\Omega} \mathrm{P_{LM}}\left(\mathrm{f_c} \bullet \mathrm{n} - \mathrm{f_v} \bullet \mathrm{n}\right) ds \approx \sum_{j\in N(i)} \frac{1}{2}\left(\mathrm{P}_{ij}\left(\mathrm{f_c} \bullet \mathrm{n}\right)\left(\mathrm{W}_i\right) + \mathrm{P}_{ij}\left(\mathrm{f_c} \bullet \mathrm{n}\right)\left(\mathrm{W}_j\right)\right) - D_{ij}\left(\mathrm{W}\right)$$

$$- \mathrm{P}_{ij}\left(\mathrm{f_v} \bullet \mathrm{n}\right)\left(\mathrm{W}_i, \mathrm{W}_j, \mathrm{grad}\left(\mathrm{W}_i\right), \mathrm{grad}\left(\mathrm{W}_i\right)\right)$$

$$\Leftrightarrow \sum_{j\in N(i)} \frac{1}{2} \mathrm{P}_{ij}\left[\left(\left(\mathrm{f_c} \bullet \mathrm{n}\right)\left(\mathrm{W}_i\right) + \left(\mathrm{f_c} \bullet \mathrm{n}\right)\left(\mathrm{W}_j\right)\right) - \mathrm{P}_{ij}^{-1} D_{ij}\left(\mathrm{W}\right) - \mathrm{f_v} \bullet \mathrm{n}\right]$$

$$\underbrace{\frac{d}{dt}\int_{\Omega_i} \mathrm{W}_i dx + \sum_{j\in N(i)} \frac{1}{2} \mathrm{P}_{ij}\left[\left(\left(\mathrm{f_c} \bullet \mathrm{n}\right)\left(\mathrm{W}_i\right) + \left(\mathrm{f_c} \bullet \mathrm{n}\right)\left(\mathrm{W}_j\right)\right) - \mathrm{P}_{ij}^{-1} D_{ij}\left(\mathrm{W}\right) - \mathrm{f_v} \bullet \mathrm{n}\right] = 0, i = 1,\ldots,N}_{\text{System of N equations}}$$

DLR